Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a homogeneous polynomial of degree d. Put $\omega = \frac{1}{k}(x_1^k, \ldots, x_n^k)$, where k is either $d+1$ if d is odd or $d+2$ if d is even. Let $f_\pm = \pm f - \omega$ and $\chi(A)$ denote Euler characteristic of A. According to [1, 2], we have

Theorem 1. *Let f be homogeneous polynomial, let f_+, f_- be as above, let $A_\pm = \{x \in S^{n-1} |\ \pm f \geq 0\}$ and $L = S^{n-1} \cap f^{-1}(0)$. Then*

  (i) f_+, f_- *have an algebraically isolated critical point at 0,*

  (ii) $\chi(A_-) = 1 - \deg_0(\nabla f_+),\ \chi(A_+) = 1 - \deg_0(\nabla f_-),$
  $\chi(L) = \chi(S^n) - \deg_0(\nabla f_+) - \deg_0(\nabla f_-),$

  (iii) *if d is odd then* $\chi(L) = \chi(S^n) - 2\deg_0(\nabla f_+).$

Example 2. Let $f = x_2^2 x_3^2 + x_1^4 - x_3^2 x_1^2$, of course f is homogeneous and $d = 4$. Then $\omega = \frac{1}{6}(x_1^6 + x_2^6 + x_3^6)$, $\nabla f_+ = (4x_1^3 - 2x_1 x_3^2 - x_1^5, 2x_2 x_3^2 - x_2^5, -2x_1^2 x_3 + 2x_2^2 x_3 - x_3^5)$ and $\nabla f_- = (-4x_1^3 + 2x_1 x_3^2 - x_1^5, 2x_2 x_3^2 - x_2^5, -2x_1^2 x_3 + 2x_2^2 x_3 - x_3^5)$. It is easy to check that $\chi(A_-) = 4$ and $\chi(L) = 0$, so

$$\deg_0(\nabla f_+) = -3 \quad \text{and} \quad \deg_0(\nabla f_-) = 3.$$

Let I denote the ideal generated by $4x_1^3 - 2x_1 x_3^2 - x_1^5, 2x_2 x_3^2 - x_2^5, -2x_1^2 x_3 + 2x_2^2 x_3 - x_3^5$ and J — generated by $-4x_1^3 + 2x_1 x_3^2 - x_1^5, 2x_2 x_3^2 - x_2^5, -2x_1^2 x_3 + 2x_2^2 x_3 - x_3^5$. Then $\deg_0(\nabla f_+)$ is equal to Poincaré-Hopf index of the ideal I, and $\deg_0(\nabla f_-)$ is equal to Poincaré-Hopf index of the ideal J. Those indexes can be computed using SINGULAR's library `phindex.lib`, and command `PH_ais`. According to SINGULAR we have that

```
> ring r=0, (x_1,x_2,x_3), ds;
> LIB "phindex.lib";
> ideal I=4*x_1^3-2*x_1*x_3^2-x_1^5,2*x_2*x_3^2-x_2^5,
-2*x_1^2*x_3+2*x_2^2*x_3-x_3^5;
> PH_ais(I);
4
> ideal J=-4*x_1^3+2*x_1*x_3^2-x_1^5,2*x_2*x_3^2-x_2^5,
-2*x_1^2*x_3+2*x_2^2*x_3-x_3^5;
> PH_ais(J);
-4
```

Example 3. Let $f = x_1^2 x_2^2 x_3^2$, then f is homogeneous polynomial of degree $d = 6$. Put $\omega = \frac{1}{8}(x_1^8 + x_2^8 + x_3^8)$, then $\nabla f_+ = (2x_1 x_2^2 x_3^2 - x_1^7, 2x_1^2 x_2 x_3^2 - x_2^7, 2x_1^2 x_2^2 x_3 - x_3^7)$ and $\nabla f_- = (-2x_1 x_2^2 x_3^2 - x_1^7, -2x_1^2 x_2 x_3^2 - x_2^7, -2x_1^2 x_2^2 x_3 - x_3^7)$. It is easy to check that $\chi(A_-) = -6$ and $\chi(L) = -6$, so

$$\deg_0(\nabla f_+) = 7 \quad \text{and} \quad \deg_0(\nabla f_-) = -1.$$

Let I denote the ideal generated by $2x_1 x_2^2 x_3^2 - x_1^7, 2x_1^2 x_2 x_3^2 - x_2^7, 2x_1^2 x_2^2 x_3 - x_3^7$ and J — generated by $-2x_1 x_2^2 x_3^2 - x_1^7, -2x_1^2 x_2 x_3^2 - x_2^7, -2x_1^2 x_2^2 x_3 - x_3^7$. Then

$\deg_0(\nabla f_+)$ is equal to Poincaré-Hopf index of the ideal I, and $\deg_0(\nabla f_-)$ is equal to Poincaré-Hopf index of the ideal J. Those indexes can be computed using SINGULAR'S library `phindex.lib`, and command `PH_ais`. According to SINGULAR we have that

```
> ring r=0, (x_1,x_2,x_3), ds;
> LIB "phindex.lib";
> ideal I=2*x_1*x_2^2*x_3^2-x_1^7,2*x_1^2*x_2*x_3^2-x_2^7,
2*x_1^2*x_2^2*x_3-x_3^7;
> PH_ais(I);
5
> ideal J=-2*x_1*x_2^2*x_3^2-x_1^7,-2*x_1^2*x_2*x_3^2-x_2^7,
-2*x_1^2*x_2^2*x_3-x_3^7;
> PH_ais(J);
-5
```

References

[1] Z. Szafraniec, *Topological Inwariants of Real Analitic Sets*, Uniwersytet Gdański, 1993,

[2] Z. Szafraniec, *Topological invariants of weighted homogeneous polynomials*, Glesgow Math. J.33 (1991) 241-245.