A. Geometric Background

Es ist die Freude an der Gestalt
in einem höheren Sinne,
die den Geometer ausmacht.

Alfred Clebsch

In this appendix we introduce a few concepts of algebraic geometry in order to illustrate some of the algebraic constructions used in this book. These illustrations are meant to stimulate the reader to develop his own geometric intuition on what is going on algebraically. Indeed, the connection between algebra and geometry has turned out to be very fruitful and both merged to become one of the leading areas in mathematics: algebraic geometry.

In order to provide some geometric understanding, we present, in this appendix, the very basic concepts of classical affine and projective varieties over an algebraically closed field K (say $\mathbb{C}$). However, we also introduce the modern counterparts, Spec and Proj, which bridge, quite naturally and with less assumptions, the canyon between algebra and geometry.

One word about the role of the ground field K. Algebraic geometers usually draw real pictures, think about them as complex varieties and perform computations over some finite field. We recommend following this attitude, which is justified by successful practice. Moreover, the modern language of schemes even allows one to formulate and prove geometric statements for arbitrary fields which coincide with the classical picture if the field is algebraically closed (cf. A.5).

For a deeper study of algebraic geometry, we recommend the book of Hartshorne [97], which is not only the standard reference book but also represents modern algebraic geometry in an excellent way, complemented perhaps by the red book of Mumford [145], and the books of Harris [95] and Brieskorn and Knörrer [25].

A.1 Introduction by Pictures

The basic problem of algebraic geometry is to understand the set of solutions $x = (x_1, \dots, x_n) \in K^n$ of a system of polynomial equations

$$f_1(x_1, \dots, x_n) = 0\,, \ \dots\,,\ f_k(x_1, \dots, x_n) = 0\,,$$

$f_i \in K[x] = K[x_1, \dots, x_n]$ and K a field. The solution set is called an algebraic set or algebraic variety. The pictures in Figures A.1 – A.8 show examples of algebraic varieties.

However, algebraic sets really live in different worlds, depending on whether K is algebraically closed or not. For instance, the question whether the simple polynomial equation $x^n + y^n + z^n = 0$, $n \geq 3$, has any non–trivial solution in K, is of fundamental difference if we ask this for K to be $\mathbb{C}$, $\mathbb{R}$ or $\mathbb{Q}$. (For $\mathbb{C}$ we obtain a surface, for $\mathbb{R}$ we obtain a surface if n is odd but only $\{0\}$ if n is even, and for $\mathbb{Q}$ this is Fermat's problem, solved by A. Wiles in 1994.) Classical algebraic geometry assumes K to be algebraically closed. Real algebraic geometry is a field in its own and the study of varieties over $\mathbb{Q}$ belongs to arithmetic geometry, a merge of algebraic geometry and number theory. In this appendix we assume K to be algebraically closed.

Many of the problems in algebra, in particular, computer algebra, have a geometric origin. Therefore, we choose an introduction by means of some pictures of algebraic varieties, some of them being used to illustrate subsequent problems.

The pictures in this introduction, Figures A.1 – A.8, were not only chosen to illustrate the beauty of algebraic geometric objects but also because these varieties have had some prominent influence on the development of algebraic geometry and singularity theory.

The Clebsch cubic itself has been the object of numerous investigations in global algebraic geometry, the Cayley and the D_4–cubic also, but, moreover, since the D_4–cubic deforms, via the Cayley cubic, to the Clebsch cubic, these first three pictures illustrate deformation theory, an important branch of (computational) algebraic geometry.

The ordinary node, also called A_1–singularity (shown as a surface singularity), is the most simple singularity in any dimension. The Barth sextic illustrates a basic but very difficult and still (in general) unsolved problem: to determine the maximum possible number of singularities on a projective variety of given degree.

Whitney's umbrella was, at the beginning of stratification theory, an important example for the two Whitney conditions. We use the umbrella in Chapter 3 to illustrate that the algebraic concept of normalization may even lead to a parametrization of a singular variety, an ultimate goal in many contexts, especially for graphical representations. In general, however, such a parametrization is not possible, not even locally, if the variety has dimension larger than one. For curves, on the other hand, the normalization is always smooth and, hence, provides, at least locally, a parametrization. Indeed, computing the normalization of the ideal given by the implicit equations for the space curve in Figure A.8, we obtain the given parametrization. Conversely, the equations are derived from the parametrization by eliminating the variable t. Elimination of variables is perhaps the most important basic application of Gröbner bases.

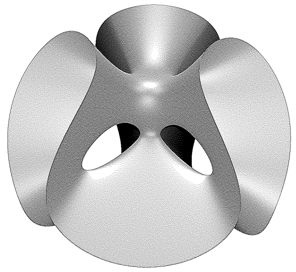


Fig. A.1. The Clebsch Cubic

This is the unique cubic surface which has $\mathbb{S}_5$, the symmetric group of five letters, as symmetry group. It is named after its discoverer Alfred Clebsch and has the affine equation

$$\begin{aligned}&81(x^3+y^3+z^3)-189(x^2y+x^2z\\&+xy^2+xz^2+y^2z+yz^2)+54xyz\\&+126(xy+xz+yz)-9(x^2+y^2+z^2)\\&-9(x+y+z)+1=0\,.\end{aligned}$$

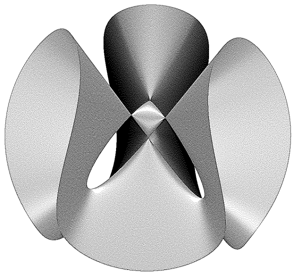


Fig. A.2. The Cayley Cubic

There is a unique cubic surface which has four ordinary nodes (see Fig. A.5), usually called the Cayley cubic after its discoverer, Arthur Cayley. It is a degeneration of the Clebsch cubic, has $\mathbb{S}_4$ as symmetry group, and the projective equation is

$$\begin{aligned}&z_0z_1z_2+z_0z_1z_3+z_0z_2z_3\\&\quad+z_1z_2z_3=0\,.\end{aligned}$$

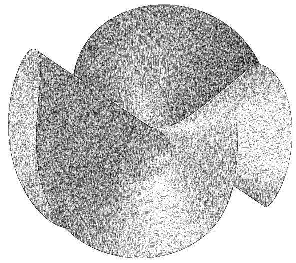


Fig. A.3. A cubic with a D_4–singularity.

Degenerating the Cayley cubic we get a D_4–singularity. The affine equation is

$$\begin{aligned}&x(x^2-y^2)+z^2(1+z)\\&\quad+\tfrac{2}{5}xy+\tfrac{2}{5}yz=0\,.\end{aligned}$$

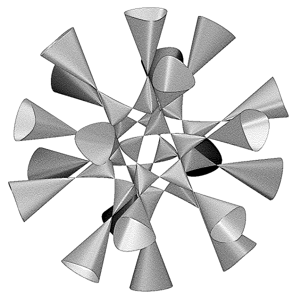


Fig. A.4. The Barth Sextic.

The equation for this sextic was found by Wolf Barth. It has 65 ordinary nodes, the maximal possible number for a sextic. Its affine equation is (with $c=\frac{1+\sqrt{5}}{2}$)

$$\begin{aligned}&(8c+4)x^2y^2z^2-c^4(x^4y^2+y^4z^2+x^2z^4)\\&\quad+c^2(x^2y^4+y^2z^4+x^4z^2)\\&\quad-\frac{2c+1}{4}(x^2+y^2+z^2-1)^2=0\,.\end{aligned}$$

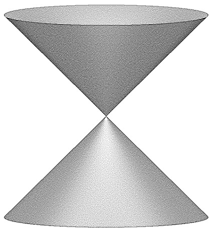


Fig. A.5. An ordinary node.

An ordinary node is the most simple singularity. It has the equation

$$x^2 + y^2 - z^2 = 0\,.$$

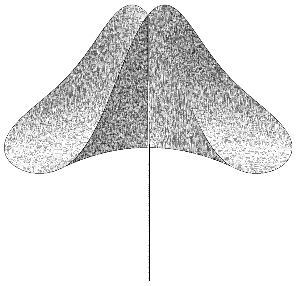


Fig. A.6. Whitney's Umbrella.

The Whitney umbrella is named after Hassler Whitney who studied it in connection with the stratification of analytic spaces. It has the affine equation

$$x^2 y - z^2 = 0\,.$$

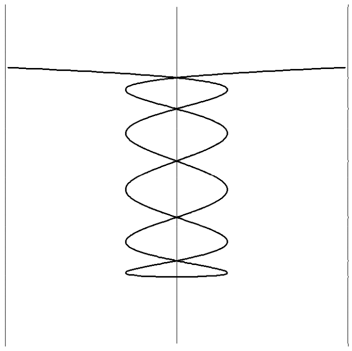


Fig. A.7. A 5–nodal plane curve of degree 11.

Deforming an A_{10}–singularity (normal form: $y^{11} - x^2 = 0$) we obtain a 5–nodal plane curve of degree 11:

$$\begin{aligned} 32x^2 &- 2097152y^{11} + 1441792y^9 \\ &- 360448y^7 + 39424y^5 \\ &- 1760y^3 + 22y - 1\,. \end{aligned}$$

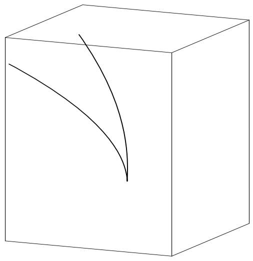


Fig. A.8. A space curve.

This space curve is given parametrically by $x = t^4,\ y = t^3,\ z = t^2$, or implicitly by

$$x - z^2 = y^2 - z^3 = 0\,.$$

Finally, the 5–nodal plane curve illustrates the singularities of plane curves, in particular, the deformation of a curve singularity into a nodal curve. Moreover, this kind of deformations, with the maximal number of nodes, also play a prominent role in the local theory of singularities. For instance, from this real picture we can read off the intersection form and, hence, the monodromy of the singularity A_{10} by the beautiful theory of A'Campo and Gusein-Zade. By means of standard bases in local rings, there exists a completely different, algebraic algorithm to compute the monodromy [161].

SINGULAR can be used to draw real pictures of plane curves or of surfaces in 3–space, that is, of hypersurfaces defined by polynomials in two or three variables. For this, SINGULAR calls the programme `surf` written by Stephan Endraß, which is distributed together with SINGULAR, but which can also be used as a stand–alone programme (unfortunately, up to now, `surf` runs only under Linux and Sun–Solaris). Drawing "nice" real pictures depends very much on the chosen equation, for example, the scaling of the variables, and the kind of projection chosen by the graphics system. It is, therefore, recommended to experiment with the concrete equations and with the parameters within `surf`. Here is the SINGULAR input for drawing the Whitney umbrella and a surprise surface:

SINGULAR Example A.1.1 (surface plot).

```
LIB "surf.lib";
ring r = 0,(x,y,z),dp;

poly f = -z2+yx2;                 //the Whitney umbrella
map phi = r,x,y,z-1/4x-2;
plot(phi(f));

f =(2*x^2+y^2+z^2-1)^3-(1/10)*x^2*z^3-y^2*z^3;
     // A surprise surface (equation due to Tore Norstrand)
phi = r,1/6z,1/6x,1/6y;           //rescaling
plot(phi(f));
```

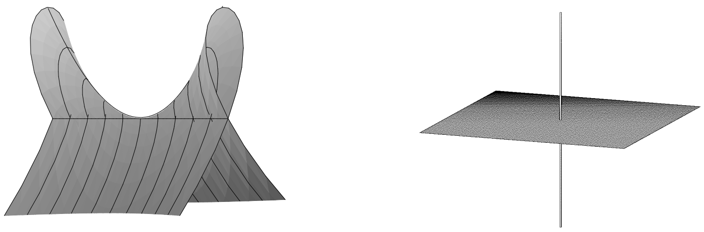
Let us now discuss some geometric problems for which the book describes algebraic algorithms, which are implemented in SINGULAR.

Recall first the most basic, but also most important, applications of Gröbner bases to algebraic constructions (Sturmfels called these "Gröbner basics"). These can be found in Chapters 1, 2 and 5 of this book.

- Ideal (respectively module) membership problem,
- Intersection with subrings (elimination of variables),
- Intersection of ideals (respectively submodules),
- Zariski closure of the image of a map,
- Solvability of polynomial equations,
- Solving polynomial equations,
- Radical membership,
- Quotient of ideals,
- Saturation of ideals,
- Kernel of a module homomorphism,

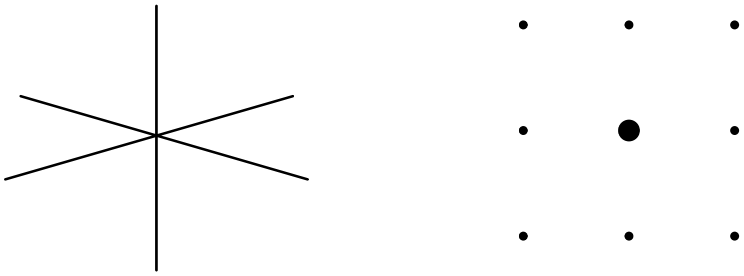
- Kernel of a ring homomorphism,
- Algebraic relations between polynomials,
- Hilbert polynomial of graded ideals and modules.

The next questions and problems lead to algorithms which are slightly more (some of them much more) involved. They are, nevertheless, still very basic and quite natural. We should like to illustrate them by means of four simple examples, shown in the Figures A.9 and A.10, referred to as Examples (A) – (D). We recommend redoing the computations using SINGULAR, the appropriate commands can be found in the chapters we refer to.



(A) The Hypersurface $V(x^2+y^3-t^2y^2)$. (B) The Variety $V(xz,yz)$.

Fig. A.9. Examples (A) and (B).



(C) The Space Curve $V(xy,xz,yz)$. (D) The Set of Points $V(y^4-y^2, xy^3-xy, x^3y-xy, x^4-x^2)$.

Fig. A.10. Examples (C) and (D).

Assume we are given an ideal $I \subset K[x] = K[x_1,\ldots,x_n]$, by a finite set of generators $f_1,\ldots,f_k \in K[x]$. Consider the following questions and problems:

(1) *Is $V(I)$ irreducible or may it be decomposed into several algebraic varieties? If so, find its irreducible components. Algebraically this means*

to compute a primary decomposition of I or of $\sqrt{I}$, the latter means to compute the associated prime ideals of I.

Example (A) is irreducible; Example (B) has two components (one of dimension 2 and one of dimension 1); Example (C) has three (one–dimensional), and Example (D) has nine (zero–dimensional) components. Primary decomposition is treated in Chapter 4.

(2) *Is I a radical ideal (that is, $I = \sqrt{I}$)? If not, compute its radical $\sqrt{I}$.*

In Examples (A) – (C) the ideal I is radical, while in Example (D) we have $\sqrt{I} = \langle y^3 - y, x^3 - x\rangle$, which is much simpler than I. In this example the central point corresponds to $V(\langle x,y\rangle^2)$ which is a *fat point*, that is, it is a solution of I of multiplicity ($= \dim_K K[x,y]/\langle x,y\rangle^2$) larger than 1 (equal to 3). All other points have multiplicity 1, hence, the total number of solutions (counted with multiplicity) is 11. This is a typical example of the kind B. Buchberger (respectively W. Gröbner) had in mind at the time of writing his thesis, [28]. We show in Chapter 4, Section 4.5, how to compute the radical. In Corollary 5.3.17 we show how to compute the dimension as a K–vector space, respectively a K–basis, of $K[x_1,\dots,x_n]/I$ if the quotient is finite dimensional.

(3) *A natural question to ask is, how independent are the generators $f_1,\dots,f_k$ of I? That is, we ask for all relations $(r_1,\dots,r_k) \in K[x]^k$ such that*

$$\sum_{i=1}^{k} r_i f_i = 0\,.$$

These relations form a submodule of $K[x]^k$, which is called the *syzygy module* of $f_1,\dots,f_k$ and is denoted by $\mathrm{syz}(I)$.[1] It is the kernel of the $K[x]$–linear module homomorphism

$$K[x]^k \longrightarrow K[x]\,,\ \ (r_1,\dots,r_k) \longmapsto \sum_{i=1}^{k} r_i f_i\,.$$

Syzygies are introduced and computed in Chapter 2, Section 2.5.

(4) *More generally, we may ask for generators of the kernel of a $K[x]$–linear map $K[x]^r \to K[x]^s$ or, in other words, for solutions of a system of linear equations over $K[x]$.*

A direct geometric interpretation of syzygies is not so clear, but there are instances where properties of syzygies have important geometric consequences, cf. [160]. To compute the kernel of a module homomorphism, see Chapter 2, Section 2.8.7.

[1] In general, the notion $\mathrm{syz}(I)$ is a little misleading, because the syzygy module depends on the chosen system of generators for I, see Chapter 2, Remark 2.5.2.

In Example (A) $\mathrm{syz}(I)=0$, in Example (B) $\mathrm{syz}(I)=\langle(-y,x)\rangle\subset K[x,y,z]^2$, in Example (C) $\mathrm{syz}(I)=\langle(-z,y,0),(-z,0,x)\rangle\subset K[x,y,z]^3$, and in Example (D) $\mathrm{syz}(I)\subset K[x,y,z]^4$ is generated by the vectors $(x,-y,0,0)$, $(0,0,x,-y)$ and $(0,x^2-1,-y^2+1,0)$.

(5) *A more geometric question is the following. Let $V(I')\subset V(I)$ be a subvariety. How can we describe $V(I)\smallsetminus V(I')$? Algebraically, this amounts to finding generators for the ideal quotient*[2]

$$I:I'=\{f\in K[x]\mid fI'\subset I\}\,.$$

Geometrically, $V(I:I')$ is the smallest variety containing $V(I)\smallsetminus V(I')$ which is the (Zariski) closure of $V(I)\smallsetminus V(I')$, a proof of this statement is given in Chapter 1, Section 1.8.9.

In Examples (B), (C) we compute the ideal quotients $\langle xz,yz\rangle:\langle x,y\rangle=\langle z\rangle$ and $\langle xy,xz,yz\rangle:\langle x,y\rangle=\langle z,xy\rangle$, which give, in both cases, equations for the complement of the z–axis $\{x=y=0\}$. In Example (D) we obtain $I:\langle x,y\rangle^2=\langle y(y^2-1),x(x^2-1),(x^2-1)(y^2-1)\rangle$, the corresponding zero–set being eight points, namely $V(I)$ without the central point.

See Chapter 1, Sections 1.8.8 and 1.8.9 for further properties of ideal quotients and for methods on how to compute them.

(6) *Geometrically important is the projection of a variety $V(I)\subset K^n$ onto a linear subspace K^{n-r}. Given generators $f_1,\dots,f_k$ of $I\subset K[x_1,\dots,x_n]$, we want to find generators for the (closure of the) image of $V(I)\subset K^n$ in $K^{n-r}=\{x\mid x_1=\dots=x_r=0\}$. The closure of the image is defined by the ideal $I\cap K[x_{r+1},\dots,x_n]$, and finding generators for this intersection is known as eliminating $x_1,\dots,x_r$ from $f_1,\dots,f_k$.*

Projecting the varieties of Examples (A) – (C) to the (x,y)–plane is, in the first two cases, surjective and in the third case it gives the two coordinate axes in the (x,y)–plane. This corresponds to the fact that the intersection with $K[x,y]$ of the first two ideals is $\langle 0\rangle$, while the third one is $\langle xy\rangle$.

Projecting the nine points of Example (D) to the x–axis we obtain, by eliminating y, the polynomial $x^2(x-1)(x+1)$, describing the three image points. This example is discussed further in Example A.3.13. The geometric background of elimination is discussed in detail in A.2 and A.3. The algorithmic and computational aspects are presented in Chapter 1, Section 1.8.2.

(7) Another problem is related to the Riemann singularity removable theorem, which states that a function of a complex manifold, which is holomorphic and bounded outside a subvariety of codimension 1, is actually holomorphic everywhere. This statement is well–known for open subsets of $\mathbb{C}$. In higher dimensions there exists a second singularity removable theorem, which states that a function, which is holomorphic outside a

[2] The same definition applies if I,I' are submodules of $K[x]^k$.

subvariety of codimension 2 (no assumption on boundedness), is holomorphic everywhere. For singular complex algebraic varieties this is not true in general, but those for which the two removable theorems hold are called *normal*. Moreover, each reduced variety has a normalization, and there is a morphism with finite fibres from the normalization to the variety, which is an isomorphism outside the singular locus.

Given a variety $V(I) \subset K^n$, the problem is to find a normal variety $V(J) \subset K^m$ and a polynomial map $K^m \to K^n$ inducing the normalization map $V(J) \to V(I)$. It can be reduced to irreducible varieties (but need not be, as shown in Chapter 3, Section 3.6), and then the equivalent algebraic problem is to find the normalization of $K[x_1, \dots, x_n]/I$, that is, the integral closure of $K[x]/I$ in the quotient field of $K[x]/I$, and to present this ring as an affine ring $K[y_1, \dots, y_m]/J$ for some m and J.

For Examples (A) – (C) it can be shown that the normalization is smooth. In (B) and (C), it is actually the disjoint union of the smooth components. The corresponding rings are $K[x_1, x_2]$, $K[x_1, x_2] \oplus K[x_3]$, $K[x_1] \oplus K[x_2] \oplus K[x_3]$. (Use SINGULAR as in Chapter 3, Section 3.6.) The fourth example (D) has no normalization, as it is not reduced.

A related problem is to find, for a non–normal variety V, an ideal H such that $V(H)$ is the non–normal locus of V.

The normalization algorithm is described in Chapter 3, Section 3.6. There, we also present an algorithm to compute the non–normal locus.

In the examples above, the non–normal locus is equal to the singular locus.

(8) The significance of *singularities* appears not only in the normalization problem. The study of singularities is also called *local algebraic geometry* and belongs to the basic tasks of algebraic geometry. Nowadays, singularity theory is a whole subject on its own (cf. A.9 for a short introduction). A singularity of a variety is a point which has no neighbourhood in which the Jacobian matrix of the generators has constant rank.

In Example (A) the whole t–axis is singular, in the other three examples only the origin.

One task is to compute generators for the ideal of the singular locus, which is itself a variety. This is just done by computing subdeterminants of the Jacobian matrix, if there are no components of different dimensions. In general, however, we need, additionally, to compute either an equidimensional decomposition or annihilators of Ext groups.

For how to compute the singular locus see Chapter 5, Section 5.7.

In Examples (A) – (D), the singular locus is given by the ideals $\langle x, y\rangle$, $\langle x, y, z\rangle$, $\langle x, y, z\rangle$, $\langle x, y\rangle^2$, respectively.

(9) *Studying a variety* $V(I)$, $I = \langle f_1, \ldots, f_k \rangle$, *locally at a singular point, say the origin of* K^n, *means studying the ideal* $IK[x]_{\langle x \rangle}$, *generated by* I *in the local ring*

$$K[x]_{\langle x \rangle} = \left\{ \frac{f}{g} \;\middle|\; f, g \in K[x],\ g \notin \langle x_1, \ldots, x_n \rangle \right\}.$$

In this local ring the polynomials g *with* $g(0) \neq 0$ *are units, and* $K[x]$ *is a subring of* $K[x]_{\langle x \rangle}$.

Now all the problems we considered above can be formulated for ideals in $K[x]_{\langle x \rangle}$ *and modules over* $K[x]_{\langle x \rangle}$ *instead of* $K[x]$.

The geometric problems should be interpreted as properties of the variety in a neighbourhood of the origin, or more generally, the given point.

At first glance, it seems that computation in the localization $K[x]_{\langle x \rangle}$ requires computation with rational functions. It is an important fact that this is not necessary, but that basically the same algorithms which were developed for $K[x]$ can be used for $K[x]_{\langle x \rangle}$. This is achieved by the choice of a special ordering on the monomials of $K[x]$ where, loosely speaking, the monomials of lower degree are considered to be larger. A systematic study is given in Chapter 1.

In A.8 and A.9 we give a short account of local properties of varieties and of singularities.

All the above problems have algorithmic and computational solutions, which use, at some place, Gröbner basis methods. Moreover, algorithms for most of these have been implemented quite efficiently in several computer algebra systems. SINGULAR is also able to handle, in addition, local questions systematically.

A.2 Affine Algebraic Varieties

From now on, we always assume K to be an algebraically closed field, except when we specify K otherwise.

We start with the simplest algebraic varieties, affine varieties.

Definition A.2.1. $\mathbb{A}^n = \mathbb{A}^n_K$ denotes the *n–dimensional affine space* over K, the set of all n–tuples $x = (x_1, \ldots, x_n)$ with $x_i \in K$, together with its structure as affine space.

A set $X \subset \mathbb{A}^n_K$ is called an *affine algebraic set* or a *(classical) affine algebraic variety* or just an *affine variety* (over K) if there exist polynomials $f_\lambda \in K[x_1, \ldots, x_n]$, λ in some index set Λ, such that

$$X = V\big((f_\lambda)_{\lambda \in \Lambda}\big) = \{x \in \mathbb{A}^n_K \mid f_\lambda(x) = 0,\ \ \forall\, \lambda \in \Lambda\}.$$

X is then called the *zero–set* of $(f_\lambda)_{\lambda \in \Lambda}$.