Preface

> In theory there is no difference
> between theory and practice.
> In practice there is.
>
> Yogi Berra

A SINGULAR Introduction to Commutative Algebra offers a rigorous introduction to commutative algebra and, at the same time, provides algorithms and computational practice. In this book, we do not separate the theoretical and the computational part. Coincidentally, as new concepts are introduced, it is consequently shown, by means of concrete examples and general procedures, how these concepts are handled by a computer. We believe that this combination of theory and practice will provide not only a fast way to enter a rather abstract field but also a better understanding of the theory, showing concurrently how the theory can be applied.

We exemplify the computational part by using the computer algebra system SINGULAR, a system for polynomial computations, which was developed in order to support mathematical research in commutative algebra, algebraic geometry and singularity theory. As the restriction to a specific system is necessary for such an exposition, the book should be useful also for users of other systems (such as *Macaulay2* and *CoCoA*) with similar goals. Indeed, once the algorithms and the method of their application in one system is known, it is usually not difficult to transfer them to another system.

The choice of the topics in this book is largely motivated by what we believe is most useful for studying commutative algebra with a view toward algebraic geometry and singularity theory. The development of commutative algebra, although a mathematical discipline in its own right, has been greatly influenced by problems in algebraic geometry and, conversely, contributed significantly to the solution of geometric problems. The relationship between both disciplines can be characterized by saying that algebra provides rigour while geometry provides intuition.

In this connection, we place computer algebra on top of rigour, but we should like to stress its limited value if it is used without intuition.

During the past thirty years, in commutative algebra, as in many parts of mathematics, there has been a change of interest from a most general theoretical setting towards a more concrete and algorithmic understanding. One of the reasons for this was that new algorithms, together with the development of fast computers, allowed non–trivial computations, which had been intractable before. Another reason is the growing belief that algorithms can contribute to a better understanding of a problem. The human idea of "understanding", obviously, depends on the historical, cultural and technical status of the society and, nowadays, understanding in mathematics requires more and more algorithmic treatment and computational mastering. We hope that this book will contribute to a better understanding of commutative algebra and its applications in this sense.

The algorithms in this book are almost all based on Gröbner bases or standard bases. The theory of Gröbner bases is by far the most important tool for computations in commutative algebra and algebraic geometry. Gröbner bases were introduced originally by Buchberger as a basis for algorithms to test the solvability of a system of polynomial computations, to count the number of solutions (with multiplicities) if this number is finite and, more algebraically, to compute in the quotient ring modulo the given polynomials. Since then, Gröbner bases have played an important role for any symbolic computations involving polynomial data, not only in mathematics. We present, right at the beginning, the theory of Gröbner bases and, more generally, standard bases, in a somewhat new flavour.

Synopsis of the Contents of this Book

From the beginning, our aim is to be able to compute effectively in a polynomial ring as well as in the localization of a polynomial ring at a maximal ideal. Geometrically, this means that we want to compute globally with (affine or projective) algebraic varieties and locally with its singularities. In other words, we develop the theory and tools to study the solutions of a system of polynomial equations, either globally or in a neighbourhood of a given point.

The first two chapters introduce the basic theories of rings, ideals, modules and standard bases. They do not require more than a course in linear algebra, together with some training, to follow and do rigorous proofs. The main emphasis is on ideals and modules over polynomial rings. In the examples, we use a few facts from algebra, mainly from field theory, and mainly to illustrate how to use Singular to compute over these fields.

In order to treat Gröbner bases, we need, in addition to the ring structure, a total ordering on the set of monomials. We do not require, as is the case in usual treatments of Gröbner bases, that this ordering be a well–ordering. Indeed, non–well–orderings give rise to local rings, and are necessary for a computational treatment of local commutative algebra. Therefore, we introduce, at an early stage, the general notion of localization. Having this, we introduce the notion of a (weak) normal form in an axiomatic way. The standard basis algorithm, as we present it, is the same for any monomial ordering, only the normal form algorithm differs for well–orderings, called global orderings in this book, and for non–global orderings, called local, respectively mixed, orderings.

A standard basis of an ideal or a module is nothing but a special set of generators (the leading monomials generate the leading ideal), which allows the computation of many invariants of the ideal or module just from its leading monomials. We follow the tradition and call a standard basis for a global ordering a Gröbner basis. The algorithm for computing Gröbner bases is Buchberger's celebrated algorithm. It was modified by Mora to compute standard bases for local orderings, and generalized by the authors to arbitrary (mixed) orderings. Mixed orderings are necessary to generalize algorithms (which use an extra variable to be eliminated later) from polynomial rings to local rings. As the general standard basis algorithm already requires slightly more abstraction than Buchberger's original algorithm, we present it first in the framework of ideals. The generalization to modules is then a matter of translation after the reader has become familiar with modules. Chapter 2 also contains some less elementary concepts such as tensor products, syzygies and resolutions. We use syzygies to give a proof of Buchberger's criterion and, at the same time, the main step for a constructive proof of Hilbert's syzygy theorem for the (localization of the) polynomial ring. These first two chapters finish with a collection of methods on how to use standard bases for various computations with ideals and modules, so–called "Gröbner basics".

The next four chapters treat some more involved but central concepts of commutative algebra. We follow the same method as in the first two chapters, by consequently showing how to use computers to compute more complicated algebraic structures as well. Naturally, the presentation is a little more condensed, and the verification of several facts of a rather elementary nature are left to the reader as an exercise.

Chapter 3 treats integral closure, dimension theory and Noether normalization. Noether normalization is a cornerstone in the theory of affine algebras, theoretically as well as computationally. It relates affine algebras, in a controlled manner, to polynomial algebras. We apply the Noether normalization to develop the dimension theory for affine algebras, to prove the Hilbert Nullstellensatz and E. Noether's theorem that the normalization of an affine ring (that is, the integral closure in its total ring of fractions) is a finite extension. For all this, we provide algorithms and concrete examples on how to compute them. A highlight of this chapter is the algorithm to compute the non–normal locus and the normalization of an affine ring. This algorithm is based on a criterion due to Grauert and Remmert, which had escaped the computer algebra community for many years, and was rediscovered by T. de Jong. The chapter ends with an extra section containing some of the larger procedures, written in the Singular programming language.

Chapter 4 is devoted to primary decomposition and related topics such as the equidimensional part and the radical of an ideal. We start with the usual, short and elegant but not constructive proof, of primary decomposition of an ideal. Then we present the constructive approach due to Gianni, Trager and Zacharias. This algorithm returns the primary ideals and the associated primes of an ideal in the polynomial ring over a field of characteristic 0, but also works well if the characteristic is sufficiently large, depending on the given ideal. The algorithm, as implemented in SINGULAR is often surprisingly fast. As in Chapter 3, we present the main procedures in an extra section.

In contrast to the relatively simple existence proof for primary decomposition, it is extremely difficult to actually decompose even quite simple ideals, by hand. The reason becomes clear when we consider the constructive proofs which are all quite involved, and which use many non–obvious results from commutative algebra, field theory and Gröbner bases. Indeed, primary decomposition is an important example, where we learn much more from the constructive proof than from the abstract one.

In Chapter 5 we introduce the Hilbert function and the Hilbert polynomial of graded modules together with its application to dimension theory. The Hilbert polynomial, respectively its local counterpart, the Hilbert–Samuel polynomial, contains important information about a homogeneous ideal in a polynomial ring, respectively an arbitrary ideal, in a local ring. The most important one, besides the dimension, is the degree in the homogeneous case, respectively the multiplicity in the local case. We prove that the Hilbert (–Samuel) polynomial of an ideal and of its leading ideal coincide, with respect to a degree ordering, which is the basis for the computation of these functions. The chapter finishes with a proof of the Jacobian criterion for affine K–algebras and its application to the computation of the singular locus, which uses the equidimensional decomposition of the previous chapter; other algorithms, not using any decomposition, are given in the exercises to Chapter 7.

Standard bases were, independent of Buchberger, introduced by Hironaka in connection with resolution of singularities and by Grauert in connection with deformation of singularities, both for ideals in power series rings. We introduce completions and formal power series in Chapter 6. We prove the classical Weierstraß preparation and division theorems and Grauert's generalization of the division theorem to ideals, in formal power series rings. Besides this, the main result here is that standard bases of ideals in power series rings can be computed if the ideal is generated by polynomials. This is the basis for computations in local analytic geometry and singularity theory.

The last chapter, Chapter 7, gives a short introduction to homological algebra. The main purpose is to study various aspects of depth and flatness. Both notions play an important role in modern commutative algebra and algebraic geometry. Indeed, flatness is the algebraic reason for what the ancient geometers called "principle of conservation of numbers", as it guarantees that certain invariants behave continuously in families of modules, respectively varieties. After studying and showing how to compute Tor–modules, we use Fitting ideals to show that the flat locus of a finitely presented module is open. Moreover, we present an algorithm to compute the non–flat locus and, even further, a flattening stratification of a finitely presented module. We study, in some detail, the relation between flatness and standard bases, which is somewhat subtle for mixed monomial orderings. In particular, we use flatness to show that, for any monomial ordering, the ideal and the leading ideal have the same dimension.

In the final sections of this chapter we use the Koszul complex to study the relation between the depth and the projective dimension of a module. In particular, we prove the Auslander–Buchsbaum formula and Serre's characterization of regular local rings. These can be used to effectively test the Cohen–Macaulay property and the regularity of a local K–algebra.

The book ends with two appendices, one on the geometric background and the second one on an overview on the main functionality of the system SINGULAR.

The geometric background introduces the geometric language, to illustrate some of the algebraic constructions introduced in the previous chapters. One of the objects is to explain, in the affine as well as in the projective setting, the geometric meaning of elimination as a method to compute the (closure of the) image of a morphism. Moreover, we explain the geometric meaning of the degree and the multiplicity defined in the chapter on the Hilbert Polynomial (Chapter 5), and prove some of its geometric properties. This appendix ends with a view towards singularity theory, just touching on Milnor and Tjurina numbers, Arnold's classification of singularities, and deformation theory. All this, together with other concepts of singularity theory, such as Puiseux series of plane curve singularities and monodromy of isolated hypersurface singularities, and many more, which are not treated in this book, can be found in the accompanying libraries of SINGULAR.

The second appendix gives a condensed overview of the programming language of SINGULAR, data types, functions and control structure of the system, as well as of the procedures appearing in the libraries distributed with the system. Moreover, we show by three examples (Maple, Mathematica, MuPAD), how SINGULAR can communicate with other systems.

How to Use the Text

The present book is based on a series of lectures held by the authors over the past ten years. We tried several combinations in courses of two, respectively four, hours per week in a semester (12 – 14 weeks). There are at least four aspects on how to use the text for a lecture:

(A) Focus on computational aspects of standard bases, and syzygies.

A possible selection for a two–hour lecture is to treat Chapters 1 and 2 completely (possibly omitting 2.6, 2.7). In a four–hour course one can treat, additionally, 3.1 – 3.5 together with either 4.1 – 4.3 or 4.1 and 5.1 – 5.3.

(B) Focus on applications of methods based on standard basis, respectively syzygies, for treating more advanced problems such as primary decomposition, Hilbert functions, or flatness (regarding the standard basis, respectively syzygy, computations as "black boxes").

In this context a two–hour lecture could cover Sections 1.1 – 1.4 (only treating global orderings), 1.6 (omitting the algorithms), 1.8, 2.1, Chapter 3 and Section 4.1. A four–hour lecture could treat, in addition, the case of local orderings, Section 1.5, and selected parts of Chapters 5 and 7.

(C) Focus on the theory of commutative algebra, using SINGULAR as a tool for examples and experiments.

Here a two–hour course could be based on Sections 1.1, 1.3, 1.4, 2.1, 2.2, 2.4, 2.7, 3.1 – 3.5 and 4.1. For a four–hour lecture one could choose, additionally, Chapter 5 and Sections 7.1 – 7.4.

(D) Focus on geometric aspects, using SINGULAR as a tool for examples.

In this context a two–hour lecture could be based on Appendix A.1, A.2 and A.4, together with the needed concepts and statements of Chapters 1 and 3. For a four–hour lecture one is free to choose additional parts of the appendix (again together with the necessary background from Chapters 1 – 7).

Of course, the book may also serve as a basis for seminars and, last but not least, as a reference book for computational commutative algebra and algebraic geometry.

Working with SINGULAR

The original motivation for the authors to develop a computer algebra system in the mid eighties, was the need to compute invariants of ideals and modules in local rings, such as Milnor numbers, Tjurina numbers, and dimensions of modules of differentials. The question was whether the exactness of the Poincaré complex of a complete intersection curve singularity is equivalent to the curve being quasihomogeneous. This question was answered by an early version of SINGULAR: it is not [149]. In the sequel, the development of SINGULAR was always influenced by mathematical problems, for instance, the famous Zariski conjecture, saying that the constancy of the Milnor number in a family implies constant multiplicity [90]. This conjecture is still unsolved.

Enclosed in the book one finds a CD with folders `EXAMPLES`, `LIBRARIES`, `MAC`, `MANUAL`, `UNIX` and `WINDOWS`. The folder `EXAMPLES` contains all SINGULAR Examples of the book, the procedures and the links to Mathematica, Maple and MuPAD. The other folders contain the SINGULAR binaries for the respective platforms, the manual, a tutorial and the SINGULAR libraries. SINGULAR can be installed following the instructions in the `INSTALL_<platform>.html` (or `INSTALL_<platform>.txt`) file of the respective folder. We also should like to refer to the SINGULAR homepage http://www.singular.uni-kl.de

which always offers the possibility to download the newest version of Singular, provides support for Singular users and a discussion forum. Moreover, one finds there a lot of useful information around Singular, for instance, more advanced examples and applications than provided in this book.

Comments and Corrections

We should like to encourage comments, suggestions and corrections to the book. Please send them to either of us:

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We also encourage the readers to check the web site for *A SINGULAR Introduction to Commutative Algebra*,

http://www.singular.uni-kl.de/Singular-book.html

This site will contain lists of corrections, respectively of solutions for selected exercises.

Acknowledgements

As is customary for textbooks, we use and reproduce results from commutative algebra, usually without any specific attribution and reference. However, we should like to mention that we have learned commutative algebra mainly from the books of Zariski–Samuel [190], Nagata [146], Atiyah–Macdonald [4], Matsumura [125] and from Eisenbud's recent book [51]. The geometric background and motivation, present at all times while writing this book, were laid by our teachers Egbert Brieskorn and Herbert Kurke. The reader will easily recognize that our book owes a lot to the admirable work of the above–mentioned mathematicians, which we gratefully acknowledge.

There remains only the pleasant duty of thanking the many people who have contributed in one way or another to the preparation of this work. First of all, we should like to mention Christoph Lossen, who not only substantially improved the presentation but also contributed to the theory as well as to proofs, examples and exercises.

The book could not have been written without the system Singular, which has been developed over a period of about fifteen years by Hans Schönemann and the authors, with considerable contributions by Olaf Bachmann. We feel that it is just fair to mention these two as co–authors of the book, acknowledging, in this way, their contribution as the principal creators of the Singular system.[1]

[1] *"Software is hard. It's harder than anything else I've ever had to do."* (Donald E. Knuth)

Further main contributors to SINGULAR include: W. Decker, A. Frühbis-Krüger, H. Grassmann, T. Keilen, K. Krüger, V. Levandovskyy, C. Lossen, M. Messollen, W. Neumann, W. Pohl, J. Schmidt, M. Schulze, T. Siebert, R. Stobbe, M. Wenk, E. Westenberger and T. Wichmann, together with many authors of SINGULAR libraries mentioned in the headers of the corresponding library.

Proofreading was done by many of the above contributors and, moreover, by Y. Drozd, T. de Jong, D. Popescu, and our students M. Brickenstein, K. Dehmann, M. Kunte, H. Markwig and M. Olbermann. Last but not least, Pauline Bitsch did the LaTeX–typesetting of many versions of our manuscript and most of the pictures were prepared by Thomas Keilen.

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The book is dedicated to our families, especially to our wives Ursula and Marlis, whose encouragement and constant support have been invaluable.

Kaiserslautern, March, 2002

Gert–Martin Greuel
Gerhard Pfister

[2] The heart is displayed by using the programme `surf`, see SINGULAR Example A.1.1.

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