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Using

SINGULAR

and its Standard Bases Engine

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Objects in SINGULAR

$\longrightarrow$ are **closely related to mathematical objects !**

In particular, almost all objects exist only in a `ring`.

- **Ring data types:**

`ring`	`qring`

- **Ring dependent data types:**

`ideal`	`map`	`matrix`
`module`	`number`	`poly`
`resolution`	`vector`	

- **Ring independent data types:**

`int`	`intmat`	`intvec`
`string`		

- **Special data types:**

`def`	`link`	`list`
`proc`	`package`	

Rings in SINGULAR

$\longrightarrow$ are given by the following data:

$$\left.\begin{array}{l}\bullet \text{ the } \textbf{characteristic} \\ \bullet \text{ the } \textbf{parameters}\end{array}\right\} \text{ of the ground field},$$

$$\left.\begin{array}{l}\bullet \text{ the } \textbf{variables} \\ \bullet \text{ the } \textbf{ordering}\end{array}\right\} \text{ of (localized) polynomial ring}.$$

Basic Examples:

(1) $\mathbb{Q}[x, y]$ is implemented by, e.g.,

```
ring r = 0,(x,y),dp;
```

(2) $\mathbb{Q}[x, y]_{\langle x,y\rangle}$ is implemented by, e.g.,

```
ring r1 = 0,(x,y),ds;
```

(3) $\mathbb{Q}(a)[x, y]$ is implemented by, e.g.,

```
ring r2 = (0,a),(x,y),dp;
```

(4) $\mathbb{Q}(\sqrt{-1})[x, y]$ is implemented by, e.g.,

```
ring r3 = (0,I),(x,y),dp;
minpoly = I^2+1;
```

Rings Associated to Orderings

A **monomial ordering** is a *total* ordering $>$ on the set of monomials $x^\alpha := x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$ satisfying

$$x^\alpha > x^\beta \implies x^\gamma x^\alpha > x^\gamma x^\beta$$

for all $\alpha, \beta, \gamma \in \mathbb{Z}^n_{\geq 0}$.

$\longrightarrow$ to any such monomial ordering $>$ we associate

$$S_> := \{u \in K[x] \setminus \{0\} \mid \mathrm{LM}(u) = 1\}$$

(LM = largest monomial), and the ring

$$K[x]_> := S_>^{-1} K[x] = \left\{ \frac{f}{u} \;\middle|\; f, u \in K[x],\, \mathrm{LM}(u) = 1 \right\}.$$

$\longrightarrow$ in particular, the following are **equivalent**:

1. $K[x]_> = K[x]$.
2. $x^\alpha > 1$ for all $\alpha \neq (0, \ldots, 0)$.
3. $x_i > 1$ for $i = 1, \ldots, n$.
4. $>$ is a well-ordering .
5. $>$ refines the natural (partial) ordering $\geq_{\mathrm{nat}}$.

Then we say that the ordering is **global** (indicated by `p`).

Rings Associated to Orderings (2)

$\longrightarrow$ we say that the ordering $>$ is **local** (indicated by `s`) if one of the following equivalent conditions holds:

1. $K[x]_> = K[x]_{\langle x \rangle}$.
2. $x^\alpha < 1$ for all $\alpha \neq (0, \ldots, 0)$.
3. $x_i < 1$ for $i = 1, \ldots, n$.
4. the inverse ordering $\left(x^\alpha >' x^\beta :\Leftrightarrow x^\alpha < x^\beta\right)$ is global.

$\longrightarrow$ if the ordering is neither local nor global then we call it a **mixed** ordering. In this case:

$$K[x] \subsetneq K[x]_> \subsetneq K[x]_{\langle x \rangle} \,.$$

$\longrightarrow$ An important method to construct (mixed) orderings is given by **product orderings** $(>_1, >_2)$:

$$\begin{aligned} x^\alpha y^\beta > x^{\alpha'} y^{\beta'} \;:\Longleftrightarrow\; & x^\alpha >_1 x^{\alpha'} \\ & \text{or } \left(x^\alpha = x^{\alpha'} \text{ and } y^\beta >_2 y^{\beta'}\right). \end{aligned}$$

Examples

$\longrightarrow$ some familiar and some less familiar rings, associated to a polynomial ring and a monomial ordering: let

$$K[x,y] = K[x_1,\dots,x_n,y_1,\dots,y_m]$$

with **product ordering** $(>_1,>_2)$, where

(1) $>_1$ *global* , $>_2$ *local* :

$$K[x,y]_> = (K[y]_{\langle y\rangle})[x] = K[y]_{\langle y\rangle} \otimes_K K[x]\,.$$

(2) $>_1$ *local* , $>_2$ *global* :

$$(K[x]_{\langle x\rangle})[y] \subsetneq K[x,y]_> \subsetneq K[x,y]_{\langle x\rangle}\,,$$

(3) $>_1$ *global* , $>_2$ *arbitrary* :

$$K[x,y]_> = (K[y]_{>_2})[x]\,.$$

To implement $K[x,y]_{\langle x\rangle}$, we have to choose the ground field $K(y)$ and a *local* ordering $>$:

$$K(y)[x]_> = K[x,y]_{\langle x\rangle}\,.$$

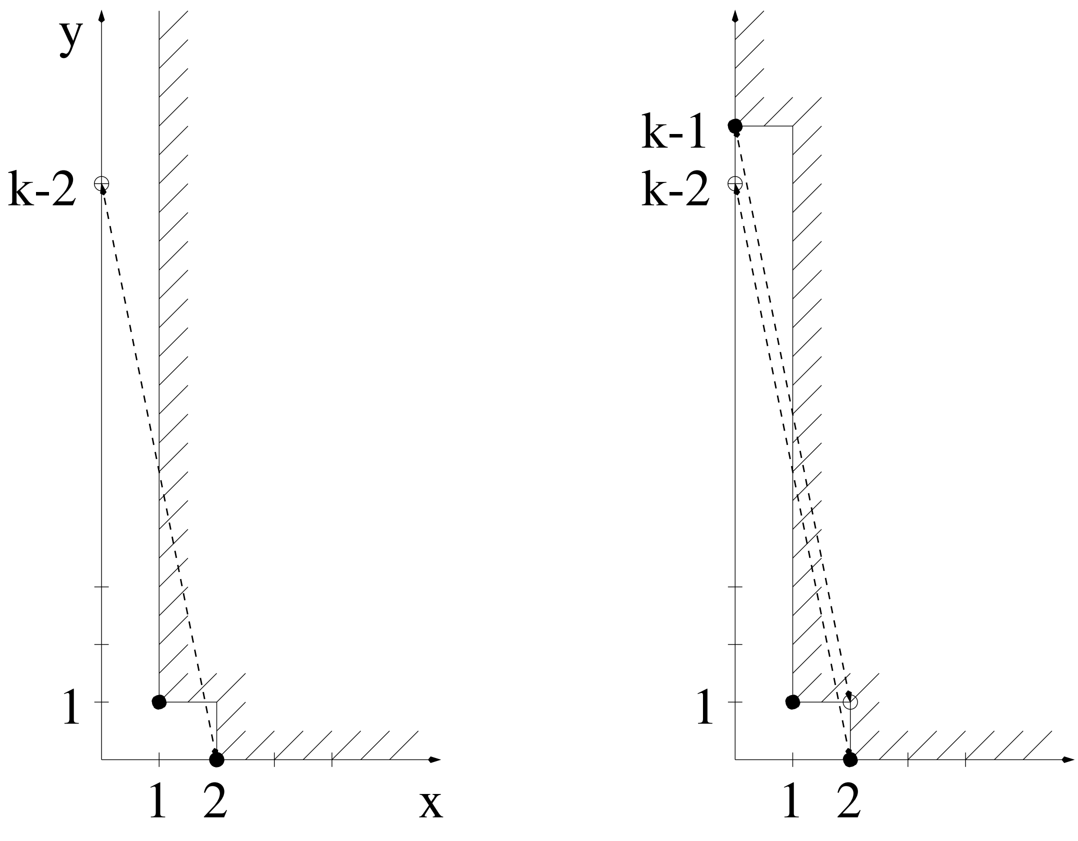
Why Standard Basis Computations ?

Problem: Compute the (local) Milnor number of

$$f = y(x^2 - y^{k-2})\,,$$

that is, compute

$$\dim_K K[x,y]_{\langle x,y\rangle} \big/ \langle\, 2xy,\; x^2 - (k-1)\cdot y^{k-2}\,\rangle\,.$$



Why Standard Basis Computations ? (2)

In general, let $I \subset K[x_1, \ldots, x_n]_>$ be an ideal. Then

$$\dim_K(K[x]_>/I) = \dim_K\big(K[x]/L(I)\big) .$$

The leading ideal

$$L(I) := \big\langle \mathrm{LM}(g) \bigm| g \in I \big\rangle$$

is generated by **monomials**, hence, to compute the codimension on the right-hand side is purely **combinatorial**.

$\longrightarrow$ We need a *special set of generators* for I:

A finite set $G \subset K[x]_>$ is called a **standard basis** of I if

- $G \subset I$,
- $L(I) = \langle \mathrm{LM}(g) \mid g \in G \rangle$.

Applications of Standard Bases

Further basic applications of standard basis computations (for ideals) are the following:

- ideal membership test ,
- (smallest) field of definition of an ideal ,
- intersection with subrings (elimination of variables) ,
- radical membership test ,
- intersection of ideals ,
- quotient of ideals ,
- saturation ,
- kernel of a ring map ,
- test for algebraic dependence ,
- subalgebra membership test ,
- solving systems of polynomial equations .

Elimination of Variables

Problem: Given $f_1, \ldots, f_k \in K[x] = K[x_1, \ldots, x_n]$,

$$I = \langle f_1, \ldots, f_k \rangle_{K[x]},$$

find generators of

$$I' = I \cap K[x_{s+1}, \ldots, x_n], \quad s < n\,.$$

Solution: (1) Choose a (global) **elimination ordering** $>$ for $x_1, \ldots, x_s$: for all $f \in K[x]$,

$$\Big(\,\mathrm{LM}(f) \in K[x_{s+1}, \ldots, x_n] \Longleftrightarrow f \in K[x_{s+1}, \ldots, x_n]\,\Big)\,.$$

Possible choices:

- lexicographical ordering `lp` (most expensive)
- product ordering `(dp(s),dp(n-s))`
- ordering with extra weight vector
- .....

(2) Compute a standard basis $S = \{g_1, \ldots, g_k\}$ of I.

(3) Those g_i, for which $\mathrm{LM}(g_i)$ does not involve $x_1, \ldots, x_s$, generate I' (they are even a standard basis of I').

Some Remarks on the SINGULAR Syntax

- SINGULAR awaits input at the prompt '`>`';
  if previous line did not end with '`;` ', at the prompt '`.`'

- Statements are terminated by a '`;`'.

- Syntax is **C-like**.

  In particular, there are

assignments	⟨variable⟩ `=` ⟨value⟩
comparisons	`==`, `<`, `>`, `!=`, etc.
conditional statements	`if`, `else`
loops	`for`, `while`
blocks	{ , }
comments	`//`, `/* */`

- Help is available by the command `help` ⟨topic⟩`;`.

- Additional functionality may be accessed by loading libraries, that is by the command `LIB` "⟨library name⟩"`;`.

Writing Procedures and Libraries

General Structure of a Library:

```
/**** Library Header ****/
version = "⟨library version⟩";
info = "⟨General help text for the Library⟩";
LIB "⟨necessary library⟩";

/****** Procedure ******/
[static]
proc ⟨procedure name⟩ (⟨parameter list⟩)
"⟨help text of the procedure⟩"
{
      procedure body
}
example {
      example of usage
}

/**** next Procedure ****/
...
```

⟶ Help and Example sections may be omitted.

⟶ Static procedures are not available for interactive use.

Debugging Tools

~; Breakpoint inside a procedure.

Singular awaits input - possibility to query value of variables. Procedure continues after ⟨**`return`**⟩ at empty line of input.

`listvar(⟨type⟩);`

List all variables of this type.

`listvar(⟨ring-name⟩);`

List all variables belonging to this ring.

`printlevel`

Raise value of integer variable `printlevel` to obtain more output from library routines.

`dbprint(⟨debuglevel⟩, ⟨text⟩);`

Print the text, if `printlevel` is at least equal to `debuglevel`.

libparse : External program to check libraries.