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Coset Enumeration using Prefix Gröbner Bases in Mrc – An Experimental Approach

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Abstract

Based on prefix Gröbner bases as implemented in Mrc 1.2 (see [13]) we study a new method to enumerate cosets. It is compared to well known methods for Todd-Coxeter enumeration using the examples from [1, 3] where studies of these methods are reported. New insights into coset enumeration were gained using three different kinds of orderings combined with the new frameworks and strategies implemented in Mrc 1.2.

Keywords

Gröbner bases, monoid and group rings, prefix reduction, coset enumeration

1 Introduction

In 1936 Todd and Coxeter [16] developed a procedure to enumerate cosets of a subgroup of a group which turned out to be one strong tool for studying groups in combinatorial group theory. Implementations of these ideas are available in computer algebra systems for groups as GAP, Cayley, and ace2. Other approaches to coset enumeration using the Knuth-Bendix algorithm or automata followed.
This paper now tries to use a new approach consisting of several frameworks and strategies to gain new insights into this process based on prefix Gröbner bases. In fact this could be narrowed down to prefix string rewriting for the examples considered here. On the other hand the examples presented by Linton in [5] can be computed using prefix Gröbner bases, too, and thus justify an extensive study of coset enumeration based on prefix Gröbner bases. As the prefix Gröbner basis setting in Mrc

1.2 produces a lot of overhead and hence we considered numbers of cosets instead of measuring times. However, if our findings prove to be valuable, they can either be incorporated into existing implementations of Todd Coxeter or if this is not possible, a specialized reimplementation on the basis of prefix string rewriting is possible. We present the results for a set of examples presented in [1, 3] which were achieved using our methods. As expected we are sometimes better, sometimes worse than the existing methods like Felsch- or HLT-type enumeration procedures. Remarkably, for some examples we are considerably better than these methods.

Further, if we compare our approach to the findings of Havas presented in [3] we get the following results. First of all, all methods including our own are syntactical ones. While the existing methods use strategies to cleverly fill the coset table our strategy introduces new cosets based on the multiplication table represented by the prefix Gröbner basis. We found, too, that adding the symmetric relators improved the enumeration process considerably. Further, our method treats all group generators as subgroup generators per se which was introduced for the Felsch method by Havas in [3].

Besides the possibility to study coset enumeration from a different point of view due to a different procedure, current methods seem to use length-lexicographical orderings with the precedence on the letters being the only parameter. Our method supports all kinds of orderings provided they are well-founded. Three different types of them, Knuth-Bendix orderings, syllable orderings and of course the length-lexicographical ordering are implemented in MRC 1.2 right now and were used to compute the examples. Results achieved using the Knuth-Bendix and the syllable orderings can be much better compared with the length-lexicographical one.

Taking for example (see also [3, 4]) the Fibonacci group $F(2,7)$ presented by $ab = c, bc = d, cd = e, de = f, fg = a, ga = b$, we get that the index of any subgroup generated by one element is 1, that is $index(F(2,7)| < f >) = 1$. This group was studied in detail in [4]. Currently (according to [3]) the best coset enumeration program enumerates 169 cosets before the collaps occurs. Using our program we can do the same enumeration by defining 134 cosets using a Knuth-Bendix ordering before the collaps occurs. This is already much closer to the target of 129 cosets set by Leech in [4]. But is still far from the best results achieved by optimizing the enumeration by hand which only needs 53 cosets before the collaps occurs. Examining more Knuth-Bendix or other orderings might improve the performance of our program further.

In this paper we show how the procedure presented in [11] for theoretical reasons to show how rewriting methods and Todd Coxeter are related, can be improved when having real computation in mind. In order to do so, we introduce so called frameworks which can be compared to the strategies Felsch and HLT for the original Todd-Coxeter enumeration procedure. They are supplemented by strategies adding cosets cleverly. In addition we studied the influence of well-founded orderings on the enumeration process and how different combinations of frameworks, strategies, and orderings behave. The improved frameworks as well as the examples evaluated are presented.

For the rest of the paper we assume that the reader is familiar with the methods available for coset enumeration. Descriptions of these methods can be found in [16, 1, 8, 3, 15]. The description of the procedure simulating the Todd-Coxeter procedure using prefix Gröbner bases and a comparison with Knuth-Bendix completion can be found in [11]. The Paper is organized as follows. In Section 2 we shortly recall the most important concept from coset enumeration and of the simulating procedure. As this procedure is only appropriate for small examples, we propose two frameworks which are presented in Section 3. They use strategies which are described in Section 4. As orderings play an important role they are introduced in Section 5. The examples used for this report are taken from [1, 3] and are presented in Section 6. The frameworks and strategies were evaluated and the results of this evaluation are described in Section 7. In Section 8 the influence of the ordering on the coset enumeration is described. Section 9 describes the results obtained for the MacDonald groups (see also [6]). Possible enhancements are pointed out in Section 11. Appendix A lists the algorithms while the results are tabulated in Appendix B, C, and D.

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2 Todd-Coxeter Simulation

The Todd-Coxeter coset enumeration is a famous method from combinatorial group theory for studying finitely presented groups. It is based on the following fundamental observations: Presenting a group $\mathcal{G}$ in terms of generators Σ and relators R corresponds to viewing it as the quotient of the free group $\mathcal{F}$ (generated by Σ) by the normal subgroup $\mathcal{N}$ generated by R. $\mathcal{N}$ can be viewed as the subgroup of $\mathcal{F}$ generated by $N(R) = \{w \circ r \circ w^{-1} \mid w \in \mathcal{F}, r \in R\}$. Notice, that if R is finite, $\mathcal{N}$, while finitely generated as a normal subgroup of $\mathcal{F}$, need not be finitely generated as a subgroup.

Now given a subgroup $\mathcal{U}$ of $\mathcal{G}$ for $g \in \mathcal{G}$ we can study the **cosets** $\mathcal{U}g = \{u \circ g \mid u \in \mathcal{U}\}$ of $\mathcal{U}$ in $\mathcal{G}$. Since for $g, h \in \mathcal{G}$ either $\mathcal{U}g = \mathcal{U}h$ or $\mathcal{U}g \cap \mathcal{U}h = \emptyset$ the group $\mathcal{G}$ is a disjoint union of cosets and the number of different cosets is called the index $|\mathcal{G} : \mathcal{U}|$ of $\mathcal{U}$ in $\mathcal{G}$. We know that if $\mathcal{U}$ is generated by a set $U \subseteq \mathcal{G}$ the index of $\mathcal{U}$ in $\mathcal{G}$ is the same as the index of the subgroup $\mathcal{H}$ generated by $U \cup N(R)$ in $\mathcal{F}$. While it is undecidable whether a subgroup has finite index in a group, coset enumeration attempts to *verify* whether the index is finite by enumerating (potential) cosets and their multiplication table. Detailed descriptions of TC procedures can be found e. g. in [16, 1, 8, 3, 15].

In [11] Procedure 1 (see page 25) was presented which simulates the Todd-Coxeter procedure to enumerate cosets using th ecomputation of prefix Gröbner bases in free group rings (see [7]). Besides the sets F_R and F_U which contain the encoded relators and subgroup relators, respectively, we have three sets. First of all, N which contains the potential coset representatives of the subgroup generated. Next, G which represents the multiplication table obtained so far and is used to decide whether or not elements of N are indeed coset representatives or not. The set B serves as a test set for possible coset representatives. Elements from this set which are not prefix reducible are added to the set N during the enumeration process. A detailed description of the meaning of the sets and of the procedure can be found in [11].

While this approach is based on the idea of computing the index of the subgroup $\mathcal{H}$ generated by $U \cup N(R)$ in $\mathcal{F}$ and hence uses Gröbner basis techniques in $\mathbb{K}[\mathcal{F}]$, a splitting of the relations R can lead to new enumeration procedures [10]. For $R = R_1 \cup R_2$ where either R_1 is complete or can be finitely completed to $\widetilde{R_1}$, let $\mathcal{G}_1$ be the group presented by $< \Sigma, R_1 >$, respectively $< \Sigma, \widetilde{R_1} >$. Then in a similar fashion we can try to compute the index of the subgroup $\mathcal{H}_1$ generated by $U \cup N(R_2)$ in $\mathcal{G}_1$ which is of course again $|\mathcal{F} : \mathcal{H}| = |\mathcal{G} : \mathcal{U}|$. The resulting procedure is nearly identical with the original procedure extended_todd_coxeter_simulation on input F_{R_2} and F_U except that now the computation takes place in $\mathbb{K}[\mathcal{G}_1]$ and we have to be more careful in choosing τ to ensure fairness.

In [10] the possibility of moving relators from the set of relators generating the group to the free group was studied. Let the free group be given as $\mathcal{F} = (\Sigma, G)$, were $G = \{(aa^{-1}, \lambda) | a \in \Sigma\} \cup \{(a^{-1}a, \lambda) | a \in \Sigma\}$. Let the relators be given by $R = \{r\}$. If one of the relators has the form $r = \sigma^n, \sigma \in \Sigma, n \in \mathbb{N}$ then we can remove this relator from R and add it to G. The latter is then completed with respect to the ordering required using the Knuth-Bendix completion procedure which is this case is always terminating. We thus get G' the completed set of rules and $R' = R \backslash \{r\}$. Using these we can perform coset enumeration as before. Other relators, too, can be moved provided a finite and convergent presentation for the underlying group exists.

The original procedure extended_todd_coxeter_simulation was implemented in MRC 1.2 (see [13]) and examples known from the literature (see [1, 3]) were computed. Soon it was clear that the original version is far too slow to be of any use except for very small examples. This is not surprising as the Gröbner basis computation which is performed after each new coset element added is quite costly (see also Section 3.3 for a comparison to known methods for coset enumeration).

Thus, different frameworks and strategies have been developed in order to find out whether the coset enumeration process described is useful compared to currently known procedures and strategies. Due to implementational reasons time and space requirements are much higher for our implementation compared to available implementations of the Todd-Coxeter-procedure. First of all, in MRC 1.2 cosets are represented by words, not by numbers, which requires more space for large examples. Second, the coefficients are elements of $\mathbb{Q}$ while they can take the values $\{-1, 0, 1\}$ only. Third, since MRC 1.2 handles polynomials, we are encoding relators as polynomials while using binomials would be sufficient. Changing the domain of the coefficients

and using binomials reduced time and space requirements drastically.
However, more important to us is whether our system can compete with respect to the crucial numbers of Todd-Coxeter-Enumeration: the maximal number of cosets defined during the enumeration process and the total number of cosets defined. Since the enumeration is performed in a different algebraic structure, new parameters play an important role and might help to study cosets from a different point of view.
All frameworks and strategies presented here can as well be used for monoids as for free and general groups (compare [9]).

3 Frameworks

As already mentioned in the previous section, computing the prefix Gröbner basis after each addition of a new coset is much to costly for all but very small examples. As a consequence it is necessary to define a certain number of cosets before recomputing the multiplication table, i.e. the prefix Gröbner basis. This might lead to the introduction of a large number of unnecessary cosets but for larger examples it is much faster than the simple approach as fewer Gröbner basis computations are necessary.
Two different frameworks have been considered which are described in Section 3.1 and 3.2. They are compared to two strategies for Todd-Coxeter enumeration in Section 3.3. The frameworks use the procedures additional_elements_start and additional_elements which add elements, which normally would not be considered during the enumeration process until much later, to the border set after the first and after each subsequent Gröbner basis computation, respectively. These are described in Section 4.
Both frameworks depend on the organization of the borderset. The borderset is currently organized as a queue with an additional data structure ordered set used to check whether elements are already in the borderset. The queue implies that always the first element of the queue is selected and new elements are added to the end. As the characters of the alphabet are added to the borderset in the same order in which they are read from the input, the elements are considered from smallest to largest with respect to a length-lexicographical ordering with the precedence small to large as read from the input. According to the algorithm, elements are skipped which can not lead to new cosets.

3.1 Steps-Framework

The first framework considered uses a value *recompute* provided by the user. After *recompute* new elements have been added to the coset table N the Gröbner basis computation is performed. Procedure 2 (see page 26) implements this framework.
Unfortunately, the choice of the parameter *recompute* is not obvious as due to undecidability results it is not known beforehand whether the number of cosets is finite

or not. Especially, it is not known how many cosets will be generated. This becomes even more complicated as there are examples where the number of cosets which have to be defined exceeds the number of cosets of the final result. Thus, a reasonably good value can only be chosen if the result (that is total and maximal number of cosets) is already known before the computation is done but in general this is impossible for unknown examples at the moment.

This framework depends on the order in which elements of the same length are considered. Changing the order in which the characters are given to the program or the way in which elements are selected from the borderset changes immediately the results unless $recompute = 1$.

3.2 Level-Framework

The "level"-framework which is implemented by Procedure 3 (see page 27) is independent of such a value. Instead of a fixed number supplied by the user, an internal condition triggers the Gröbner basis computation.

At the beginning only the letters of the alphabet are added to the border set B. This is called level 1. Each element selected from the border set B which is a new coset is added to the coset-set N. Further, it is multiplied with all letters of the alphabet except the inverse of its last letter. These Elements have length 2 and are added to the new border set B_{new}. Finally, the multiplication table G is extended. If the current border set is empty the new border set becomes current and the Gröbner basis computation is performed. Thus, the border set of level n consist of all elements of length n which were generated by the previous level $n-1$. They are examined before the Gröbner basis of G is computed again. As the recomputation of the Gröbner basis is triggered by the procedure itself this framework is self-adapting with respect to the number of new cosets defined.

As all elements of the current level n are considered before the recomputation of G, that is all elements of length n, this method is independent of the order in which these elements are considered. Nevertheless, this framework, too, still depends on the method used to select elements from the borderset.

3.3 Comparison with Todd-Coxeter Strategies

In [1, 3] two strategies for Todd-Coxeter enumeration were described in detail. Felsch-Type methods try to extract as much information as possible from the relators and the current coset table and only define new cosets if necessary. HLT and look-ahead HLT, which will be refered to as HLT in the sequel, define as many new cosets as possible and only extract new information from the relators and the coset table if necessary.

If we compare our frameworks to these strategies, it can be seen that the original procedure follows the Felsch-Type methods and has the same drawbacks: extracting information is costly. The "steps"- and the "level"-framework are both similar to HLT: define a certain number of new cosets before extracting new information. The difference is that HLT uses some fixed sized table, i.e. the number of maximal definable cosets is fixed beforehand while our frameworks limit the number of newly defined cosets and the coset table, i.e. the ideal basis, is allowed to grow.
While the "steps"-framework has a similar problem as HLT, that is, the number of cosets to be defined before extracting new information must be given before starting the computation, this is not the case for the "level"-framework.

4 Strategies

Already known coset enumeration procedures try to use "clever" strategies to select the coset to consider next. These strategies are based on information about the multiplication table.
The two frameworks would normally select the cosets in increasing order beginning with the smallest as described in Section 3. But there might be cosets which are quite large with respect to the ordering and lead to important information, e.g. that two or more cosets are identical. Using this information the enumeration process might be shortened considerably. Thus, in order to enhance the frameworks, strategies were provided to add elements to the borderset which normally would only be considered much later in the enumeration process.
It is neither obvious which elements should be considered nor when they should be added. Right now, the elements are added using the procedures additional_elements_start and additional_elements which add elements to the borderset after the first and each subsequent Gröbner basis computation, respectively, according to the following strategies:

1. NONE: No elements are added.

2. P-ALL: additional_elements_start and additional_elements add all prefixes of the head-terms of the polynomials of G: $\{w | w \in \Sigma^+ \text{ and } \exists p \in G\ \exists v \in \Sigma^* : wv \equiv \mathsf{HT}(p)\}$.

3. P-R: additional_elements_start and additional_elements add those prefixes of the head-terms of the polynomials of G which, when multiplied with the generators, could lead to polynomials which can be reduced using G: $\{u | u \in \Sigma^+ \text{ and } \exists w \in \Sigma^+ : [\exists p \in G\ \exists v \in \Sigma^* : wv \equiv \mathsf{HT}(p) \text{ and } \exists v \in \Sigma^+\ \exists z \in \Sigma^*\ \exists r \in R : uv \equiv w \wedge vz \equiv \mathsf{HT}(r)]\}$

4. P-G: additional_elements_start and additional_elements add those prefixes of the head-terms of the polynomials of G which, when multiplied with polynomials of G, could lead to polynomials which can be reduced using G: $\{u | u \in \Sigma^+$ and $\exists w \in \Sigma^+ : [\exists p \in G\ \exists v \in \Sigma^* : wv \equiv \mathsf{HT}(p)$ and $\exists v \in \Sigma^+\ \exists z \in \Sigma^*\ \exists g \in G : uv \equiv w \wedge vz \equiv \mathsf{HT}(g)]\}$

5. I-ALL: additional_elements_start and additional_elements add all inverse terms of the prefixes of the head-terms of the polynomials of G: $\{w^{-1} | w \in \Sigma^*$ and $\exists p \in G\ \exists v \in \Sigma^* : wv \equiv \mathsf{HT}(p)\}$.

6. I-R: additional_elements_start adds all inverse terms of the head-terms of the relators: $\{w^{-1} | w \in \Sigma^*$ and $\exists r \in R\ \exists v \in \Sigma^* : wv \equiv \mathsf{HT}(r)\}$. additional_elements adds no elements to the borderset.

7. I-R-P: additional_elements_start adds all inverse terms of the head-terms of the relators: $\{w^{-1} | w \in \Sigma^*$ and $\exists r \in R\ \exists v \in \Sigma^* : wv \equiv \mathsf{HT}(r)\}$. additional_elements_start and additional_elements add all prefixes of the head-terms of the polynomials of G: $\{w | w \in \Sigma^*$ and $\exists p \in G\ \exists v \in \Sigma^* : wv \equiv \mathsf{HT}(p)\}$.

8. ENUM: additional_elements_start and additional_elements add elements in a special order.

9. RANDOM: additional_elements_start and additional_elements add random elements.

Only those elements are added which are neither already in the coset-set N nor in the border set B and which are not prefix-reducible using G. This reduces the maximal size of the border set B and the new border set B_{new}, respectively.
The last two strategies for adding elements, namely ENUM and RANDOM, allow additional parameters. They are not considered further here.

5 Orderings

As will be seen later, orderings play an important rôle for the enumeration process. The strategies presented will behave differently when they are combined with different orderings. Three different orderings are defined which are well-founded, total, and admissible. They are implemented in MRC 1.2 and are used for the coset enumerations.

Definition 1 (Length-Lexicographical Ordering)
Let Σ be a finite alphabet. Let $>$ be a total precedence on Σ. Let $v \equiv v_1 \dots v_n \in \Sigma^*, v_i \in \Sigma, w \equiv w_1 \dots w_m \in \Sigma^*, w_j \in \Sigma$. The **length-lexicographical ordering** *$\succ_{ll}$ is defined as:*

$$v \succ_{ll} w \quad \textit{iff} \quad |v| > |w| \textit{ or } \\ (|v| = |w| \textit{ and } \exists\, 1 \leq i \leq n : (v_i > w_i \wedge \forall\, 1 \leq j < i : v_j \equiv w_j))$$

Definition 2 (Knuth-Bendix Ordering)
Let Σ be a finite alphabet. Let $>$ be a total precedence on Σ. Let $v \equiv v_1 \dots v_n \in \Sigma^*, v_i \in \Sigma, w \equiv w_1 \dots w_m \in \Sigma^*, w_j \in \Sigma$. Let $g : \Sigma \to \mathbb{N}$ be a total function attaching a weight to each letter of the alphabet. Then g is defined on words as:

$$\begin{aligned} g: \quad \Sigma^* &\to \mathbb{N} \\ u &\mapsto \sum_{i=1}^{|u|} g(u_i) \end{aligned}$$

The **Knuth-Bendix ordering (kbo)** *$\succ_{kbo}$ is defined as:*

$$v \succ_{kbo} w \quad \textit{iff} \quad g(v) > g(w) \textit{ or } \\ (g(v) = g(w) \textit{ and } \exists\, 1 \leq i \leq n : (v_i > w_i \wedge \forall\, 1 < j \leq i : v_j \equiv w_j))$$

Definition 3 (Syllable Ordering)
Let Σ be a finite alphabet. Let $>$ be a total precedence on Σ. Let $\tau : \Sigma \to \{l, r\}$ be a total function named **status function**. *Let $\max(u), u \in \Sigma^*$ be the largest letter of u with respect to the precedence on Σ.*
The **syllable ordering** *$\succ_{syl,\tau}$ is defined as:*

$$\begin{aligned} v \succ_{syl,\tau} w \textit{ iff } & |v|_{\max(vw)} > |w|_{\max(vw)} \textit{ or} \\ & [\ \max(vw) = a \textit{ and } |v|_a = |w|_a = n \textit{ and} \\ & \quad v \equiv v_1 a v_2 a \dots v_n a v_{n+1} \textit{ and } w \equiv w_1 a w_2 a \dots w_n a w_{n+1} \textit{ and} \\ & \quad [\ \tau(a) = r \textit{ and} \\ & \quad\ \ \exists\, 1 \leq i \leq n+1 : (v_i \succ_{syl,\tau} w_i \wedge \forall\, i+1 \leq j \leq n+1 : v_j \equiv w_j] \textit{ or} \\ & \quad [\ \tau(a) = l \textit{ and} \\ & \quad\ \ \exists\, 1 \leq i \leq n+1 : (v_i \succ_{syl,\tau} w_i \wedge \forall\, 1 \leq j \leq i-1 : v_j \equiv w_j]] \end{aligned}$$

Notice, that the syllable ordering has a recursive definition. In MRC 1.2 the orderings are implemented according to the definitions given with the restriction that for the syllable ordering $\forall\, a \in \Sigma : \tau(a) = l$ or $\forall\, a \in \Sigma : \tau(a) = r$. That is, for all letters of the alphabet syllables are either compared from left to right or from right to left.
The comparison of terms is frequently needed for example for the determination of the head term after each reduction step. Therefore, the time needed to compare two terms with respect to the chosen ordering directly influences the time needed to compute the prefix Gröbner basis. As can be seen from the definitions, the length-lexicographical ordering has at least to compare the length of the two terms and at most compare the length and compare each character of the terms. The Knuth-Bendix ordering has at least to compute the sum of both terms and at most compute the sum and compare

each character of the term. The syllable ordering has at least to count the number of syllables of each term and at most in addition compare the syllables recursively. Thus, we have that the expense needed for the comparison of two terms grows from length-lexicographical over Knuth-Bendix to syllable orderings.

6 Examples and Notations

In [1, 3], two different methods for the Todd-Coxeter enumeration process of cosets, namely Felsch- and HLT-style coset enumeration procedures, were studied and the results for some examples tabulated. We use these examples and compare the results achieved there to the results we obtained by using the frameworks and strategies presented in the previous sections. The examples are based on the groups which are given together with their presentation in Table 1 on page . The examples themselves together with the indeces are given in Table 3 (see page 28). Inverse elements are given by capital letters. The results will be summarized in the following sections and are tabulated in the appendix (see the respective sections).
We will use "n/m" to denote the number of cosets defined where n is the maximal number and m the total number of cosets defined.

7 Evaluation of the Frameworks and the Strategies

Thirteen examples were tabulated in [1, 3]. They were computed using the "level" framework introduced in Section 3 and the seven strategies 1-7 introduced in Section 4 for adding elements to the border set. Neither the "steps"-framework nor the strategies 8 and 9 were examined as they depend on additional parameters and therefore are more difficult to evaluate.
A length-lexicographical ordering (see Section 5) was used with the precedences chosen as depicted in Table 3 (see Appendix B.1, page 28). The results together with the findings of [3] are presented in Table 4 – 7 (see Appendix B.2, pages 29 – 30) . The first column shows the best results for Felsch-Type enumeration, the second the results for lookahead HLT-Style. The third column shows the results for strategy NONE while columns 4-6 show either the prefix strategies P-ALL, P-G, and P-R or the inverse strategies I-ALL, I-R, and I-R-P.
The NONE-strategy performs better than HLT for 8 out of 13 examples considering the maximal number of cosets defined, and for 10 out of 13 examples considering the total number of cosets defined. Compared to the best results of the Felsch strategy which were presented in [3] it performs better only for two examples considering maximal and total number of cosets.
Adding elements makes things worse for most of the examples considered. But there are notable exceptions. Adding the inverses of the relators (I-R) reduces the number

Group	Presentation
E_1	$TrtRR = RsrSS = StsTT = 1$
$(2,5,7;2)$	$a^2 = b^5 = (ab)^7 = [a,b]^2 = 1$
$G^{3,7,17}$	$a^3 = b^7 = c^{17} = (ab)^2 = (bc)^2 = (ca)^2 = (abc)^2 = 1$
$PSL_2(11)$	$a^{11} = b^2 = (ab)^3 = (a^4bA^5b)^2 = 1$
$(2,3,7;7)$	$a^2 = b^3 = (ab)^7 = [a,b]^7 = 1$
$M_{11}^{(1)}$	$a^{11} = b^5 = c^4 = (a^4c^2)^3 = (bc^2)^2 = (abc)^3 = BabA^4 = CbcB^2 = 1$
$(8,7\|2,3)$	$a^8 = b^7 = (ab)^2 = (Ab)^3 = 1$
Neu	$a^3 = b^3 = c^3 = (ab)^5 = (Ab)^5 = (ac)^4 = (aC)^4 = aBabCacaC = (bc)^3 = (Bc)^4 = 1$
$Cam(3)$	$r^2srsR^3S = s^2rsrS^3R = 1$
$G^{3,7,16}$	$a^3 = b^7 = c^{16} = (ab)^2 = (bc)^2 = (ca)^2 = (abc)^2 = 1$
$G(2,4)$	$BAbaBabA^2 = ABabAbaB^4 = 1$
$G(2,6)$	$BAbaBabA^2 = ABabAbaB^6 = 1$
$G(3,3)$	$BAbaBabA^3 = ABabAbaB^3 = 1$

Table 1: Presentations of the Groups used for the Examples

of cosets to be defined for $G(2,4)|E$ and $G(2,6)|E$ to about 65%.
Adding all cyclic permutations of the relators to the set of relators improves the enumeration considerably. Unfortunately, this also slows down the computation in most cases, as the Gröbner bases to be computed are larger and contain more information. The results are shown in Table 8 – 11 (see Appendix B.3, pages 31 – 32). Especially, if no elements are added to the border set we get a performance which is at least as good or even much better as without these permutations. In this case, NONE performs better than HLT with the exception of $G^{3,7,16}|E$ were the maximal number

of cosets defined is higher while there are only half as many cosets defined totally. Compared to the Felsch-strategy, 8 out of 13 examples can be computed defining less cosets. Remarkable are the MacDonald groups $G(n,m)|E$ where 3 to 7 times less cosets have to be defined.

Now, adding elements does not improve the behaviour with the following exceptions. Adding the inverses of the relators (I-R) reduces the maximal/total number of cosets to be defined for $E_1|E$ from 157/157 to 97/97 which is about about 38% less. The same strategy reduces the numbers for $Neu| < a, c >$ slightly from 1683/1697 to 1637/1671. The same behaviour is found for $G(2,4)|E$ where a reduction from 467/467 to 424/424 is achieved and for $G(3,3)|E$ where the reduction is from 9753/9753 to 9253/9253 while for $G(2,6)|E$ more cosets have to be enumerated. For $G^{3,7,17}| < ab, c >$ strategy I-R-P leads to 945/998 cosets enumerated compared to 1153/1153 using strategy NONE. For $G^{3,7,16}|E$ all other strategies perform better than NONE.

8 The Influence of the Ordering

The ordering chosen for the computation of the examples presented in the previous section is a length-lexicographical one. But this is not necessarily the best one. First of all, it is only one of several different length-lexicographic orderings possible which was selected more or less randomly. Further, two other types of orderings, namely Knuth-Bendix orderings (kbo) and two kinds of syllable orderings which were presented in Section 5, are implemented in Mrc 1.2. This allows us to study which influence the ordering has on the maximal/total number of cosets to be defined. Here, we neglect the fact, that these orderings are more complex and therefore use more time than the length-lexicographical one. As already mentioned time and space are not considered right now.

The examples were computed using different kbos, length-lexicographical and syllable orderings. For the length-lexicographical ordering and the syllable orderings at least one example was computed. The orderings together with the results are tabulated in Appendix C.

We use the following abbreviations: kbo-x means a Knuth-Bendix ordering where x has the greatest weight attached, normally all other weights are one. The abbreviation ll-ZzXxYy describes the length-lexicographical ordering with precedence $Z > z > X > x > Y > y$ while syl-l-ZzXxYy and syl-r-ZzXxYy describe the syllable orderings with the precedence being as for the length-lexicographical ordering and the l, respectively r, designating that the syllables are compared from left to right, respectively right to left.

8.1 $E_1|E$

The first example reveals two characteristics. We have that for all orderings computed the strategy I-R is better than NONE, followed by I-ALL and P-R. Further, the strategies P-ALL, P-G, P-R and I-R-P, that is all strategies involving prefixes, are equally bad. On the other hand, for all strategies the syllable orderings perform better than the Knuth-Bendix and the length-lexicographic orderings. The performance of the latter is rather similar.
Further, this example shows that the differences between different orderings for the same strategy can be quite large. If we consider the best strategy found using a length-lexicographical ordering, namely I-R, we see that the number of cosets defined ranges from 59/89 to 97/97 (maximal/total). Remarkably, the best ordering is syl-r-tsrTSR while ll-tsrTSR is the one performing worst. For I-ALL the values range from 99/99 for syl-l-tsrTSR to 547/547 for ll-tsrTSR. Table 12 lists the orderings while Table 13 lists the results.

8.2 $(2, 5, 7; 2)|E$

This example shows that there can be large differences for one strategy and almost none for others. Using the strategy NONE, all orderings seem to perform equally well, the maximal/total number of cosets defined ranging from 143/143 to 163/163 (115%). For P-G the maximal/total number of cosets defined ranges from 127/139 for kbo-b to 276/287 (217%/194%) for ll-BbAa. Remarkably, the strategy P-G together with the ordering kbo-b performs best while together with the ordering ll-BbAa it performs worst compared to all other combinations computed.
Compared to the first example, here we have no strategy performing better than the others for all orderings or one ordering being better than the others for all strategies. The best orderings with respect to the strategies are kbo-A, kbo-a, and ll-BbAa for NONE, kbo-B for P-ALL, kbo-b for P-G, kbo-b for P-R, ll-BbAa for I-ALL, syl-r-abAB for I-R, and syl-l-abAB for I-R-P. The best strategy with respect to the orderings are NONE for kbo-A, P-ALL for kbo-B, NONE for kbo-a, P-G for kbo-b, I-ALL for ll-BbAa, I-R-P for syl-l-abAB, and I-R for syl-r-abAB.

8.3 $G^{3,7,17}| < ab, c >$

The third example differs from the first two as the subgroup is not E but $< ab, c >$. At a first glance, adding inverse elements does not lead to any difference. But for ll-CcbaBA the number of cosets enumerated differs between I-ALL and NONE, and for both length-lexicographical orderings there are differences between I-R and NONE and between I-ALL and NONE in the enumeration sequence. Further P-ALL and I-R-P give similar results for all orderings but the length-lexicographic ones. This is not surprising as we get $c = C = \lambda$ and $a = B$ and $b = A$ almost immediately from the subgroup relators. As in the previous example, no strategy is best for all orderings nor is one ordering best for all strategies.

8.4 $PSL_2(11)|E$

In this example, the maximal number of cosets defined equals the final number of cosets for most combinations. Thus, only the total number of cosets defined is of interest for most strategies. Strategy NONE reveals to be best with respect to all orderings for this example, the best combinations being NONE together with kbo-A and kbo-a. The number of totally defined cosets differs only slightly ranging from 667 to 684.

For the other strategies there is a greater variation and no distinct order which one performs best.

8.5 $(2, 3, 7; 7)|E$

This coxeter group shows another feature of coset enumeration. No combination of strategy and ordering is best with respect to the maximal and the total number of cosets defined. With respect to the maximal number of cosets defined the combination P-R and kbo-A (1105/1566) is best while with respect to the total number of cosets defined we get the best result for the combination P-ALL and kbo-b (1368/1449). There are other combinations which reveal the same behaviour. Overall, no strategy or ordering is better or worse than the others.

8.6 $M_{11}^{(1)}| < a >$

This example has two features: its subgroup is generated by a and only as many cosets are defined as the result has. As for $PSL_2(11)|E$ the total number of cosets is interesting.

Here, strategy NONE with the length-lexicographical ordering is optimal because exactly 720 cosets are defined totally. Note, that this ordering performs worst for I-ALL.

8.7 $(8, 7|2, 3)| < a^2, a^{-1}b >$

This example gives best results for the strategy NONE using kbo-A. The values for NONE and I-ALL are the same for all orderings except the length-lexicographical one. Here, too, for different strategies different orderings are optimal.

8.8 $Neu| < a, c >$

The same holds for this example, for different strategies different orderings are optimal.

8.9 $Cam(3)|E$

In this example, we have two winners, strategy NONE and ll-SsRr with 161/173 and syl-l-RSrs with 131/233 (maximal/total number of cosets defined) depending on which of the two values is considered more important. The orderings kbo-R and kbo-r with 158/211 and kbo-S and kbo-s with 153/207 lie between those two and might be preferred. All other strategies perform slightly (factor 1.5) or considerably (factor 10) worse.

8.10 $G^{3,7,16}|E$

Strategy P-R together with the ordering kbo-B enumerates 43703/52149 cosets (maximal/total) and thus performs best of all combinations computed. Compared with the best Felsch method it still performs about two times worse.

8.11 $G(3,3)|E$

Here, all prefix strategies (P-ALL, P-G, P-R, and I-R-P) perform worse or considerably worse than NONE. Remarkably, I-ALL performs best together with the syllable orderings while being up to 7 times worse than NONE for length-lexicographical orderings. Strategy I-R performs as good as or better than NONE. Further, it depends on the strategy which of the two syllable-left orderings performs better.

8.12 Summary

The examples show that the combination of all parameters has to be considered to find good enumerations. While some examples always exhibit the best result for one strategy regardless of the ordering chosen, others depend on the right combination of both. In addition, we have to choose whether we want a faster execution defining as few cosets as possible or if we want to use as little space as possible meaning to get a minimal number of maximal defined cosets.

9 The MacDonald Groups

Having evaluated the combination of several strategies and several orderings for the examples considered, it became clear that strategy and ordering have to be chosen individually for each example. Now, one question is whether there exist families of examples whose members behave uniformly. We will explain the concept in Section 9.1. In the Sections 9.2 – 9.4 we will examine how the Macdonald groups behave with respect to this concept.

9.1 Families of Groups

The question raised in the introduction is whether there exist families of examples whose members behave uniformly. That is, given such a family one can select a combination of one strategy and one ordering which is optimal for all members of this family. While it is not quite probable that such a family exists there might exist families whose members are similar enough that the selected strategy and ordering are pretty good for all the members while not being the best.

If such families exist then one could try to learn how to enumerate cosets for members of this family efficiently using the following procedure. Compute some 1000 combinations for a small member, pick those performing best plus some randomly chosen, and compute larger examples using these combinations. Of course, this is only a good idea if the computation of such a large set is feasible and one of the following two situations is met. First, if a large problem is intractable using the available resources. Then finding an optimal solution for a smaller example seems the best way to find a combination which makes it possible to treat the large one. Second, if more than one problem of a family is to be solved, using the best combinations can reduce the total time to find the solutions. Otherwise, it would be more reasonable to compute the examples directly choosing different combinations of strategies and orderings by hand.

The Macdonald groups $G(n, m)$ (see also [6]) which form a syntactic family were analyzed in order to see if they are a family with respect to coset enumeration, too. They are generated by $\{BAbaBabA^n, ABabAbaB^m\}$ and seem to behave better in the following sense. They all have finite index though only a very bad approximation of the index is known:

$$\mathsf{index}(G(n,m)) \quad | \quad 27 \cdot (n-1) \cdot (m-1) \cdot (gcd(n,m))^8$$

But for $n = 2$ the following holds:

$$\forall m : \mathsf{index}(G(2,m)) = m - 1$$

9.2 The Macdonald Groups $G(2, m)$

We performed the coset enumeration of the groups $G(n, m), n = 2, m = 2, \ldots, 11, 15$ in order to see whether they behave uniformly with respect to coset enumeration as implemented in Mrc 1.2 or not. Using the "level"-framework (see Section 3) seven strategies (NONE, P-ALL, P-G, P-R, I-ALL, I-R, I-R-P: see Section 4) and seven orderings (kbo-A, kbo-B, kbo-a, kbo-b, ll-BbAa, syl-l-BbAa, syl-r-BbAa: see Section 5 & 8) were combined and the results were compared. Overall, P-ALL, P-G and I-R-P perform almost equally worst. That is, why they were not further considered. The results are tabulated in Appendix D on page 46ff.

For $m = 2$, the results for the orderings do not differ much for each of the strategies selected. The strategy NONE is the best with 17 maximal and 17 totally defined cosets. The next one is I-R with 42/42 to 47/47 maximal/total defined cosets. Then comes I-ALL with 46/46 to 71/71 and finally P-R with 49/49 to 71/71, I-ALL being better than P-R for all orderings computed except ll-BbAa were they are equal. Remarkably, no combination of strategies and orderings defines more cosets totally than maximal.

For $m = 3$, NONE and I-R perform almost equally good with less than 161 cosets defined, I-R being better for all orderings but syllable-right. The two other strategies define between 300 and 650 cosets, except for I-ALL together with the syllable-right ordering which defines 171/192 cosets. This is still worse than for NONE and I-R. I-ALL performs better than P-R for the kbo-A, kbo-a, kbo-b, length-lexicographic and syllable-right ordering.

From this point onwards the performance of I-ALL depends very much on the ordering with kbo-A, kbo-a, and the length-lexicographical ordering being bad, and kbo-B and kbo-b being good while the behaviour of the syllable orderings is changing.

For $m = 4$, the picture is different. While P-R is worst, and I-R better than NONE except for the syllable-right ordering, I-ALL is best for both syllable orderings with respect to the total number of cosets defined while being close to the other two with respect to the maximal number of cosets defined. For kbo-B and kbo-b, I-ALL performs better than NONE.

For $m = 5$, the picture changes again. Now, NONE and I-R perform almost equally, this time NONE being better for most orderings, I-R only being better than NONE for kbo-b. For kbo-a, the length-lexicographical ordering and the syllable orderings they perform almost equally good. P-R is still worst except for the length-lexicographical ordering and the syllable-left ordering where I-ALL is worst. Remarkably, I-ALL is best for syl-r-BbAa.

For $m = 6$, we get some kind of chaos. For kbo-A, kbo-B, and ll-BbAa strategy NONE is best while for the others I-R is best. For the latter NONE performs considerably worse. I-ALL performs better than I-R for kbo-B and better than NONE for kbo-b, else worse than both. This time, P-R performs better than I-ALL for the length-lexicographical and the syllable orderings but still worse than the other strategies.

For $m = 7$, we get a similar picture, but there are the following differences. NONE performs better than I-R for kbo-a. Both perform worse for syllable-left which performed good for $m = 6$. For the syllable orderings NONE, I-ALL and P-R perform equally bad. I-ALL and P-R perform almost equally bad for kbo-A and kbo-a.

For $m = 8$, we have the following differences to $m = 7$. NONE performs better than I-ALL for kbo-b but still worse than I-R. P-R performs better than I-ALL for kbo-A and kbo-a but considerably worse for kbo-b.
For $m = 9, 10, 11$ no more changes are observed for the kbos and the length-lexicographical ordering while the syllable orderings are still slowly changing with all strategies being rather close together. Remarkably, for $m = 10, 11$ I-ALL together with syl-l-BbAa is worst with at least about two times as many cosets defined compared to all other combinations.
Thus for $m = 8, \ldots 11, 15$ we get the following picture. For kbo-A, kbo-a, and ll-BbAa we have that NONE is best, followed by I-R with 50% more cosets defined, then come P-R and finally I-ALL with about 400 % more cosets defined than NONE. The gaps are increasing with increasing m. For kbo-B we have that NONE is better than I-ALL with 50 % more cosets defined, followed by I-R with 75 % more cosets defined and finally P-R with 200 % more cosets defined. For kbo-b I-R is best with 25 % less cosets defined than NONE which defines about the same number of cosets for all kbo orderings. As for kbo-B I-ALL is next with about 50 % more cosets defined than NONE and finally we have I-R with 450 % more cosets defined. While the picture stays the same for all $m = 8, \ldots 11$, we have that for NONE the number of cosets defined does only increase between 1 to 15 cosets for increasing m. For the other strategies the number of cosets increases between 50 and 2000 for increasing m. Finally, there are the two syllable orderings. Here the fact just mentioned comes into play. While for syl-l-BbA I-R is best for $m = 8, \ldots 11$ it gets nearer to NONE with increasing m. For $m = 15$ NONE is best followed by I-R, P-R, and I-ALL while we had the order I-R, I-ALL, P-R, and finally NONE for $m = 8$. About the same with a different order holds for syl-r-BbAa.

9.3 The Macdonald Group $G(2, 8)$ with different orderings

This behaviour is rather discouraging as good combinations for small m become bad ones for large m and vice versa. It looks like, though, that from $m = 8$ on, the behaviour gets stable. Thus choosing $m = 8$ as starting point for larger examples seems to be a good idea. Therefore $G(2, 8)$ was computed using the strategies NONE, P-R, I-ALL, and I-R combined with 625 kbos and all length-lexicographic (24) and syllable-left (24 as well) orderings. Thus a total of 2792 combinations was computed and evaluated.
For the length-lexicographical orderings all strategies behave pretty uniform. Strategy NONE enumerates 1343/1343 cosets maximal/totally for all, I-R between 1718/1719 and 1780/1780, P-R between 5261/5261 and 5340/5340, and I-ALL between 6725/6725 and 6869/6869 cosets. Notice, that the maximal number of cosets defined is equal to the total number of cosets defined for all combinations, except some combinations with I-R were we have defined one additional coset.

For the kbos there is a greater variation of the results. For the strategy NONE we have that between 1210/1210 and 1343/1343 cosets were enumerated maximal/totally. Best performed those with the following weights: $(a\ 5), (b\ 1), (A\ 5), (B\ 1)$, worst those with equal weights for all letters which is in fact a length-lexicographical ordering which is used as tie breaker for kbos. That is, all kbos performed equally or better than the length-lexicographical ordering. The strategy I-R showed a greater variation. Between 1032/1033 and 2357/2359 cosets were enumerated maximal/totally. Best were those kbos attaching a high weight to b and low weights to the other letters of the alphabet. The best combination of weights was $(a\ 1), (b\ 5), (A\ 1), (B\ 1)$ in contrast to strategy NONE. The worst combinations have high weights attached to A and B and low weights to a, and b. Strategy I-ALL yielded the following results: between 2067/2067 and 7235/7235 cosets were enumerated maximal/totally. This strategy was better than I-R for some kbos. It also had the largest variation. Good kbos were those with a high weight attached to B and low weights to the other letters of the alphabet. Worst combinations were those attaching high weights to a and A and low weights to b and B. Finally, strategy P-R enumerated between 3776/3776 and 6948/6948 cosets maximal/totally together with kbos. It has a smaller variation than I-ALL. Good kbos were those with a high weight attached to B and low weights to the other letters of the alphabet as for I-ALL. But in contrast bad kbos were those attaching high weights to a and b and low weights to A and B, just the opposite to strategy NONE.

Finally for the syllable orderings we got the following results. Contrary to the length lexicographical orderings which did not differ very much from each other the syllable orderings show great veriations. For NONE we have that the best ordering is syl-l-AbBa with 1817/1845 cosets defined maximal/totally, while syl-l-abBA is worst with 12748/14743 cosets defined. For I-R the best ordering is syl-l-bBAa with 1453/1660 cosets defined, while syl-l-AaBb is worst with 13272/14788 cosets defined. For I-ALL the best ordering is syl-l-baAB with 1700/1862 cosets defined, while syl-l-bAaB is worst with 18447/20704 cosest defined and syl-l-bBAa was not computed at all. Finally for P-R the best ordering is syl-l-BbaA with 2389/2399 cosets defined, while syl-l-AbBa is worst with 4847/4849. Strategy P-R showed the slightest variation while for the other strategies the worst syllable orderings are up to 10 times worse than the best one. Further, non syllable ordering is very good or very bad for alll strategies. It should be noted, too, that the best ordering for I-R is second best for NONE and vice versa.

To sum up the results, we have that the Knuth-Bendix orderings are best for strategy NONE followed by the length-lexicographical orderings. All syllable orderings are considerably worse for this strategy. Strategy I-R shows a similar behaviour, but this time the length-lexicographical orderings enumerate as much cosets as the Knuth-Bendix orderings on average, that is about half of them enumerate fewer cosets while the other half enumerates more cosets. Most syllable orderings enumerate more cosets than the Knuth-Bendix orderings. As for strategy I-ALL, we have that almost all

Knuth-Bendix orderings perform better than the length-lexicographical orderings. Nevertheless, the best and worst orderings are a syllable orderings. Finally, for strategy P-R we have similar to I-R that the length-lexicographical orderings enumerate as much cosets as the Knuth-Bendix orderings on average, but other Knuth-Bendix orderings are best. All syllable orderings are better than the length-lexicographical ordering and most better than the Knuth-Bendix orderings.

9.4 The Macdonald Groups $G(3, m)$

Unfortunately, the groups obtained for $n = 3$ already seem to be much harder. While the results for $G(3, 3)$ are still very good compared to Felsch or HLT style methods it was not yet possible to compute $G(3, 4)$. Here, a tuned version has to be used which uses less space. However, the results for $n = 3, m = 3$ show some similarities to the results obtained for $n = 2$. The strategies NONE and I-R perform best for most orderings used followed by I-ALL which is even the best for the syllable ordering syl-l-BbAa. But I-ALL is considerably worse for the length-lexicographical orderings. All other strategies perform even worse producing the best results for the syllable orderings. The best combination was I-ALL/syl-l-BAab which was not used for $n = 2$. Here, further investigations are necessary.
It should nevertheless be noted that $G(3, 3), G(3, -1)$, and $G(-1, -1)$ are all isomorphic to the generalized quaternion group of order 16 and that the coset enumeration for the latter two is almost trivial!

10 Coset Enumeration over General Groups

In Section 2 we explained, that instead of studying the index of the subgroup generated by $U \cup N(R)$ in $\mathcal{F}$ for $R = R_1 \cup R_2$ we can instead study the index of the subgroup generated by $U \cup N(R_2)$ in $\mathcal{G}_1$ where $\mathcal{G}_1$ is the group generated by $< \Sigma, R_1 >$, respectively $< \Sigma, \widetilde{R_1} >$. This is done in this section by choosing relators from R, i.e. splitting $R = R_1 \cup R_2$, and then trying to complete R_1 with respect to the ordering chosen using the Knuth-Bendix completion procedure. When this can be done successfully, Gröbner basis techniques in more general group rings can be applied. For example, provided the relators are of the form σ^n, where $\sigma \in \Sigma$ and $n \in \mathbb{N}$, we can use the Knuth-Bendix completion procedure to get a finite, convergent presentation of the underlying group. For the examples $(2, 5, 7; 2)|E$, $G^{3,7,17}| < ab, c >$, $PSL_2(11)|E$, $(2, 3, 7; 7)|E$, $M_{11}^{(1)}| < a >$, $(2, 8|2, 3)| < a^2, a^{-1}b >$, $Neu| < a, c >$ and $G^{3,7,16}|E$ all relators having this form were used to define the new respective groups $\mathcal{G}_1$ (see Table 2). The results for the modified coset enumeration using completed group presentations for the respective $< \Sigma, R_1 >$ are tabulated in Appendix E. Notice, that each ordering may yield a different complete set of relators $\widetilde{R_1}$ for a given set of relators

Group	R_1	R_2
$(2,5,7;2)$	$a^2 = b^5 = 1$	$(ab)^7 = [a,b]^2 = 1$
$G^{3,7,17}$	$a^3 = b^7 = c^{17} = 1$	$(ab)^2 = (bc)^2 = (ca)^2 = (abc)^2 = 1$
$PSL_2(11)$	$a^{11} = b^2 = 1$	$(ab)^3 = (a^4bA^5b)^2 = 1$
$(2,3,7;7)$	$a^2 = b^3 = 1$	$(ab)^7 = [a,b]^7 = 1$
$M_{11}^{(1)}$	$a^{11} = b^5 = c^4 = 1$	$(a^4c^2)^3 = (bc^2)^2 = (abc)^3 = BabA^4 = CbcB^2 = 1$
$(8,7\|2,3)$	$a^8 = b^7 = 1$	$(ab)^2 = (Ab)^3 = 1$
Neu	$a^3 = b^3 = c^3 = 1$	$(ab)^5 = (Ab)^5 = (ac)^4 = (aC)^4 = aBabCacaC =$ $(bc)^3 = (Bc)^4 = 1$
$G^{3,7,16}$	$a^3 = b^7 = c^{16} = 1$	$(ab)^2 = (bc)^2 = (ca)^2 = (abc)^2 = 1$

Table 2: Presentations of the Groups used for the Examples

R_1. As for all other examples computed, there is no uniform behaviour detectable. The combination of ordering, framework, and additional elements strategy influences whether the results are better or worse compared to the coset enumeration modulo the free group.
The example $(2,5,7;2)|E$ in general gives better results than before, that is, fewer cosets are enumerated. The best result is now 45/45 cosets enumerated maximal/total for syl-l-abAB and I-ALL compared to 143/143 for kbo-A and NONE.
For $G^{3,7,17}| < ab, c >$ on the other hand we get worse results, that is for most combinations more cosets have to be enumerated. The best result now is 913/1045 cosets enumerated maximal/total for syl-l-CcBbAa and P-R compared to 821/867 for kbo-a and P-ALL.
For $PSL_2(11)|E$ we get optimal results for all orderings combined with strategy NONE. All other combinations are equally good or better, with two exceptions, namely strategy I-R combined with the syllable ordrings.
For $(2,3,7;7)|E$ we get better results for most combinations but the are quite some combinations which behave worse. The best result improved to 1092/1103 cosets enumerated maximal/total for kbo-B and NONE compared to 1105/1566 for kbo-A and P-R.

For $M_{11}^{(1)}| < a >$ most combinations are worse and only a few are better. The best result which was already optimal remained the same.
For $(2,8|2,3)| < a^2, a^{-1}b >$ we have almost as many combinations which are better as those which are worse. The best result improved to 448/549 cosets enumerated maximal/total for syl-l-abAB and I-R compared to 766/773 for kbo-A and NONE.
For $Neu| < a, c >$ holds the same. The best result improved to 560/656 cosets enumerated maximal/total for syl-l-CcBbAa and NONE compared to 1637/1671 for ll-CcBbAa and I-R.
For $G^{3,7,16}|E$ again we have a mix of better and worse combinations. The best result improved to 30949/42538 cosets enumerated maximal/total for ll-abcABC and P-ALL compared to 43931/56621 for syl-l-CcBbAa and P-ALL. But it is still worse than the best results for Felsch. Nevertheless, we are about as good as the default strategy of Felsch type.
Overall, we have the same problem as before. We have no hints about which combination to choose to get good results for the coset enumeration process. Nevertheless, we have an additional parameter to further improve this process. It should be noted, that we also have the choice not to add all of the relators having the form σ^n. That is, we can move only some of them or even other relators, provided that there exists a finite and convergent presentation of the underlying group.
For example, in Appendix E.3 the results for the example $G^{3,7,17}| < ab, c >$ are tabulated but this time we split the set of relators R into $R_1 : a^3 = b^7 = 1$ and $R_2 : c^{17} = (ab)^2 = (bc)^2 = (ca)^2 = (abc)^2 = 1$. This was inspired by the fact that c is one of the generators of the subgroup. As before some combinations yielded better results some worse. The best result was 911/995 cosets enumerated maximal/total for kbo-C and P-ALL and lies between the best result for the standard enumeration, which was 821/867 for kbo-a and P-ALL, and the best result for the first modified enumeration, which is 913/1045 for syl-l-CcBbAa and P-R.

11 Enhancements

We experimented with a new procedure to enumerate cosets. This has led to two frameworks and 9 strategies of which one framework and 7 strategies were evaluated further. It was found that the examples considered do not behave uniformly. It depends on the example whether there are best strategies or orderings and if there is a great variation of the number of cosets enumerated for the combinations evaluated or not. For most examples the enumeration process behaves quite differently depending largely on the combination of strategy and ordering. Further the idea of families of groups with respect to coset enumeration was evaluated.
Several enhancements are possible now. First of all the number of examples has to be expanded to give more information about the behaviour of this procedure compared to

those already known for quite a while like HLT and Felsch type methods. To do that, the implementation was specialized in order to be able to compute larger examples as time and space requirements are very high for the more general implementation.
On the other hand there are other frameworks and other strategies possible which have to be evaluated. There are still two strategies left which allow more parameters. Further, examining the results of all available methods to enumerate cosets might lead to further improvements of the methods known or even to a brand-new method which is superior to the current ones.
Finally, the idea of families of groups has to be investigated more closely. The example considered showed a very varying behaviour with respect to coset enumeration using our procedures for the first members of the family while it seems to get stable for larger members. As the groups $PSL_2(11)$, $M_{11}^{(1)}$, and Neu are not members of families of defining relations they can not be considered for testing this concept. The groups E_1 and $Cam(3)$ are such groups, but the next member of these families could not be computed with the resources available as they are already much harder. From the examples considered in this report, there remain the coxeter groups (see [2]) which form syntactic families and which have to be analyzed more closely.

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A Procedures

```
extended_todd_coxeter_simulation
```

Given: $F_R = \{r - 1 \mid r \in R\} \neq \emptyset$, a set of binomials representing the relators.
$F_U = \{u - 1 \mid u \in U\}$, a set of binomials representing the subgroup generators.
Find: N: the cosets of the subgroup U in the group G
G: the prefix Gröbner basis representing a multiplication table

$N := \{\lambda\}$;
$B := \{a \mid a \in X \cup X^{-1}\}$;
$G :=$ prefix.groebner_basis_of_right_ideal_fg$(F_R \cup F_U)$;
while $B \neq \emptyset$ **do**
    $\tau := \mathsf{min}_<(B)$;
    $B := B \backslash \{\tau\}$;
    if τ is not prefix reducible by G
        then $N := N \cup \{\tau\}$;
            $B := B \cup \{\tau a \mid a \in (X \cup X^{-1}) \backslash \{(last(\tau))^{-1}\}\}$;
            % where $last(\tau)$ is the last letter of the word τ
            $H := \{\tau * (r - 1) \mid r - 1 \in F_R\}$;
            $G :=$ prefix.groebner_basis_of_right_ideal_fg$(G \cup H)$;
            $N := N \backslash \{w \in N \mid w \text{ is prefix reducible by } G\}$;
    endif
endwhile

Procedure 1: Extended Todd-Coxeter simulation

```
extended_todd_coxeter_simulation.steps
```

Given: $F_R = \{r - 1 \mid r \in R\} \neq \emptyset$, a set of binomials representing the relators.
$F_U = \{u - 1 \mid u \in U\}$, a set of binomials representing the subgroup generators.
recompute, value to trigger computation of the multiplication table.

Find: N: the cosets of the subgroup U in the group G
G: the prefix Gröbner basis representing a multiplication table

$N := \{\lambda\}$;
$G :=$ prefix.groebner_basis_of_right_ideal_fg$(F_R \cup F_U)$;
$B := \{a \mid a \in X \cup X^{-1}\}$;
$B := B \cup$ additional_elements_start(B, G, N);
$counter\_t := 0$;
while $B \neq \emptyset$ **do**
    $\tau :=$ select(B);
    $B := B \backslash \{\tau\}$;
    if τ is a new coset
        then $counter\_t := counter\_t + 1$;
            $N := N \cup \{\tau\}$;
            $B := B \cup \{\tau a \mid a \in (X \cup X^{-1}) \backslash \{(last(\tau))^{-1}\}\}$;
            $G := G \cup \{\tau * (r - 1) \mid r - 1 \in F_R\}$;
            if $counter\_t \geq recompute$
                then $counter\_t := 0$;
                    $G :=$ prefix.groebner_basis_of_right_ideal_fg(G);
                    $N := N \backslash \{w \in N \mid w$ is prefix reducible by $G\}$;
                    $B := B \cup$ additional_elements(B, G, N);
            endif
    endif
endwhile
if $counter\_t = 0$
    then $G :=$ prefix.groebner_basis_of_right_ideal_fg(G);
        $N := N \backslash \{w \in N \mid w$ is prefix reducible by $G\}$;
fi

Procedure 2: Extended Todd-Coxeter simulation: Framework Steps extended_todd_coxeter_simulation.level

Given: $F_R = \{r - 1 \mid r \in R\} \neq \emptyset$, a set of binomials representing the relators.
$F_U = \{u - 1 \mid u \in U\}$, a set of binomials representing the subgroup generators.

Find: N: the cosets of the subgroup U in the group G
G: the prefix Gröbner basis representing a multiplication table

```
N := {λ};
G := prefix.groebner_basis_of_right_ideal_fg(F_R ∪ F_U);
B := {a | a ∈ X ∪ X^{-1}};
B := B ∪ additional_elements_start(B, G, N);
B_new := ∅;
while B ≠ ∅ do
    τ := select(B);
    B := B\{τ};
    if τ is a new coset
        then  N := N ∪ {τ};
              B_new := B_new ∪ {τa | a ∈ (X ∪ X^{-1})\{(last(τ))^{-1}}};
              G := G ∪ {τ * (r - 1) | r - 1 ∈ F_R};
    endif
    if B = ∅
        then  B := B_new;
              B_new := ∅;
              G := prefix.groebner_basis_of_right_ideal_fg(G);
              N := N\{w ∈ N | w is prefix reducible by G};
              if B ≠ ∅
                  then  B := B ∪ additional_elements(B, G, N);
              endif
    endif
endwhile
```

Procedure 3: Extended Todd-Coxeter simulation: Framework Level

B Examples

B.1 Precedences

The following table lists the index for each of the examples computed together with the precendence on the alphabet used for the length-lexicographical ordering.

Example	Index	Precedence on Σ
$E_1 \| E$	1	$T > t > S > s > R > r$
$(2,5,7;2) \| E$	1	$B > b > A > a$
$G^{3,7,17} \| < ab, c >$	1	$C > c > B > b > A > a$
$PSL_2(11) \| E$	660	$B > b > A > a$
$(2,3,7;7) \| E$	1092	$B > b > A > a$
$M_{11}^{(1)} \| < a >$	720	$C > c > B > b > A > a$
$(8,7 \| 2,3) \| < a^2, a^{-1}b >$	448	$B > b > A > a$
$Neu \| < a, c >$	240	$C > c > B > b > A > a$
$Cam(3) \| E$	120	$S > s > R > r$
$G^{3,7,16} \| E$	21504	$C > c > B > b > A > a$
$G(2,4) \| E$	3	$B > b > A > a$
$G(2,6) \| E$	5	$B > b > A > a$
$G(3,3) \| E$	16	$B > b > A > a$

Table 3: Indeces and Precedences for the Examples Computed

B.2 Without Preprocessing

Example	Felsch	HLT	NONE	P-ALL	P-G	P-R
$E_1\|E$	98	695	584	648	648	1660
$(2,5,7;2)\|E$	216	224	205	188	187	228
$G^{3,7,17}\| < ab, c >$	724	1381	1153	1857	2528	1321
$PSL_2(11)\|E$	660	661	660	715	702	660
$(2,3,7;7)\|E$	1221	2286	1534	1883	1579	1249
$M_{11}^{(1)}\| < a >$	720	721	720	2070	1057	1130
$(8,7\|2,3)\| < a^2, a^{-1}b >$	824	1241	1298	1742	1609	2139
$Neu\| < a, c >$	2650	4553	4358	15137	8240	6386
$Cam(3)\|E$	653	2189	1222	1207	4809	3238
$G^{3,7,16}\|E$	21504	69990	75162	67023	67080	67918
$G(2,4)\|E$	3188	2973	3556	1871	8550	5087
$G(2,6)\|E$	7889	4194	10593	18795	9826	16685
$G(3,3)\|E$	25481	29007	70627	150972	88335	126650

Table 4: Maximal number of cosets defined - Prefix-Strategies

Example	Felsch	HLT	NONE	P-ALL	P-G	P-R
$E_1\|E$	104	758	584	648	648	1680
$(2,5,7;2)\|E$	216	227	205	196	200	265
$G^{3,7,17}\| < ab, c >$	761	2315	1153	2235	3473	1515
$PSL_2(11)\|E$	743	824	684	826	921	879
$(2,3,7;7)\|E$	1310	2880	1602	1951	1977	1917
$M_{11}^{(1)}\| < a >$	724	1349	801	2448	1524	1612
$(8,7\|2,3)\| < a^2, a^{-1}b >$	840	1422	1300	1817	1741	2294
$Neu\| < a, c >$	2750	7158	4403	15584	8488	6668
$Cam(3)\|E$	660	2206	1222	1238	4823	3247
$G^{3,7,16}\|E$	23702	161805	75705	82283	82351	84529
$G(2,4)\|E$	3193	3255	3560	1893	8691	5248
$G(2,6)\|E$	7893	4582	10597	19154	9947	17221
$G(3,3)\|E$	25496	31993	70859	154763	89951	129594

Table 5: Total number of cosets defined - Prefix-Strategies

Example	Felsch	HLT	NONE	I-ALL	I-R	I-R-P
$E_1\|E$	98	695	584	746	730	890
$(2,5,7;2)\|E$	216	224	205	211	150	193
$G^{3,7,17}\| < ab, c >$	724	1381	1153	1153	1153	1857
$PSL_2(11)\|E$	660	661	660	660	660	660
$(2,3,7;7)\|E$	1221	2286	1534	1743	1534	1898
$M_{11}^{(1)}\| < a >$	720	721	720	1114	720	2174
$(8,7\|2,3)\| < a^2, a^{-1}b >$	824	1241	1298	1581	1147	1744
$Neu\| < a, c >$	2650	4553	4358	21975	7027	15184
$Cam(3)\|E$	653	2189	1222	1161	2140	1960
$G^{3,7,16}\|E$	21504	69990	75162	62613	78813	66887
$G(2,4)\|E$	3188	2973	3556	12663	2274	2851
$G(2,6)\|E$	7889	4194	10593	19571	7147	3647
$G(3,3)\|E$	25481	29007	70627	111249	120529	28141

Table 6: Maximal number of cosets defined - Inverse-Strategies

Example	Felsch	HLT	NONE	I-ALL	I-R	I-R-P
$E_1\|E$	104	758	584	746	735	893
$(2,5,7;2)\|E$	216	227	205	215	151	198
$G^{3,7,17}\| < ab, c >$	761	2315	1153	1155	1153	2235
$PSL_2(11)\|E$	743	824	684	987	713	742
$(2,3,7;7)\|E$	1310	2880	1602	1872	1604	1960
$M_{11}^{(1)}\| < a >$	724	1349	801	1479	955	2550
$(8,7\|2,3)\| < a^2, a^{-1}b >$	840	1422	1300	1742	1162	1822
$Neu\| < a, c >$	2750	7158	4403	23401	7217	15639
$Cam(3)\|E$	660	2206	1222	1168	2171	1971
$G^{3,7,16}\|E$	23702	161805	75705	90200	81280	82004
$G(2,4)\|E$	3193	3255	3560	12835	2283	2880
$G(2,6)\|E$	7893	4582	10597	19754	7199	3674
$G(3,3)\|E$	25496	31993	70859	112412	121480	28503

Table 7: Total number of cosets defined - Inverse-Strategies

B.3 Adding Symmetric Relators

Example	Felsch	HLT	NONE	P-ALL	P-G	P-R
$E_1 \| E$	98	695	157	572	572	572
$(2,5,7;2) \| E$	216	224	143	276	276	275
$G^{3,7,17} \| < ab, c >$	724	1381	1153	968	1576	2206
$PSL_2(11) \| E$	660	661	660	947	888	897
$(2,3,7;7) \| E$	1221	2286	1534	1561	1518	2613
$M_{11}^{(1)} \| < a >$	720	721	720	720	720	720
$(8,7 \| 2,3) \| < a^2, a^{-1}b >$	824	1241	973	1862	1759	1325
$Neu \| < a, c >$	2650	4553	1683	3828	2699	3361
$Cam(3) \| E$	653	2189	161	1884	1884	1883
$G^{3,7,16} \| E$	21504	69990	75058	59350	59338	62715
$G(2,4) \| E$	3188	2973	467	2767	2767	2585
$G(2,6) \| E$	7889	4194	1343	4746	4746	3904
$G(3,3) \| E$	25481	29007	9753	76022	76022	69897

Table 8: Maximal number of cosets defined - Prefix-Strategies

Example	Felsch	HLT	NONE	P-ALL	P-G	P-R
$E_1 \| E$	104	758	157	572	572	572
$(2,5,7;2) \| E$	216	227	143	282	287	284
$G^{3,7,17} \| < ab, c >$	761	2315	1153	1026	1887	2492
$PSL_2(11) \| E$	743	824	684	1082	1027	1036
$(2,3,7;7) \| E$	1310	2880	1602	1650	1885	2902
$M_{11}^{(1)} \| < a >$	724	1349	720	1069	860	956
$(8,7 \| 2,3) \| < a^2, a^{-1}b >$	840	1422	975	1936	1834	1389
$Neu \| < a, c >$	2750	7158	1697	3924	2784	3455
$Cam(3) \| E$	660	2206	173	1884	1884	1883
$G^{3,7,16} \| E$	23702	161805	75453	65472	65460	62716
$G(2,4) \| E$	3193	3255	467	2767	2767	2585
$G(2,6) \| E$	7893	4582	1343	4746	4746	3904
$G(3,3) \| E$	25496	31993	9753	76025	76025	69900

Table 9: Total number of cosets defined - Prefix-Strategies

Example	Felsch	HLT	NONE	I-ALL	I-R	I-R-P
$E_1\|E$	98	695	157	542	97	572
$(2,5,7;2)\|E$	216	224	143	138	176	277
$G^{3,7,17}\| < ab, c >$	724	1381	1153	1153	1153	945
$PSL_2(11)\|E$	660	661	660	660	660	967
$(2,3,7;7)\|E$	1221	2286	1534	1725	1710	1561
$M_{11}^{(1)}\| < a >$	720	721	720	720	720	720
$(8,7\|2,3)\| < a^2, a^{-1}b >$	824	1241	973	1213	1166	1860
$Neu\| < a, c >$	2650	4553	1683	9589	1637	3869
$Cam(3)\|E$	653	2189	161	1922	386	2005
$G^{3,7,16}\|E$	21504	69990	75058	47841	65109	59115
$G(2,4)\|E$	3188	2973	467	2544	424	2770
$G(2,6)\|E$	7889	4194	1343	4517	1481	4731
$G(3,3)\|E$	25481	29007	9753	57708	9253	76276

Table 10: Maximal number of cosets defined - Inverse-Strategies

Example	Felsch	HLT	NONE	I-ALL	I-R	I-R-P
$E_1\|E$	104	758	157	542	97	572
$(2,5,7;2)\|E$	216	227	143	161	177	284
$G^{3,7,17}\| < ab, c >$	761	2315	1153	1153	1153	998
$PSL_2(11)\|E$	743	824	684	876	698	1113
$(2,3,7;7)\|E$	1310	2880	1602	1834	1757	1652
$M_{11}^{(1)}\| < a >$	724	1349	720	999	760	1056
$(8,7\|2,3)\| < a^2, a^{-1}b >$	840	1422	975	1304	1167	1933
$Neu\| < a, c >$	2750	7158	1697	10066	1671	3964
$Cam(3)\|E$	660	2206	173	1922	391	2005
$G^{3,7,16}\|E$	23702	161805	75453	71238	66091	65196
$G(2,4)\|E$	3193	3255	467	2544	424	2770
$G(2,6)\|E$	7893	4582	1343	4517	1481	4731
$G(3,3)\|E$	25496	31993	9753	57709	9253	76279

Table 11: Total number of cosets defined - Inverse-Strategies

C Computing examples using different orderings

C.1 $E_1|E$

Ordering	Precedence on Σ
kbo-R	(r 1) > (R 6) > (s 1) > (S 1) > (t 1) > (T 1)
kbo-S	(r 1) > (R 1) > (s 1) > (S 6) > (t 1) > (T 1)
kbo-T	(r 1) > (R 1) > (s 1) > (S 1) > (t 1) > (T 6)
kbo-r	(r 6) > (R 1) > (s 1) > (S 1) > (t 1) > (T 1)
kbo-s	(r 1) > (R 1) > (s 6) > (S 1) > (t 1) > (T 1)
kbo-t	(r 1) > (R 1) > (s 1) > (S 1) > (t 6) > (T 1)
kbo-tsr	(r 6) > (R 1) > (s 12) > (S 1) > (t 18) > (T 1)
ll-TtSsRr	T > t > S > s > R > r
ll-tsrTSR	t > s > r > T > S > R
syl-l-tsrTSR	t > s > r > T > S > R
syl-r-tsrTSR	t > s > r > T > S > R

Table 12: Orderings for $E_1|E$

Ordering	NONE	P-ALL	P-G	P-R	P-R-2	I-ALL	I-R	I-R-P
kbo-R	146	561	561	538	205	421	96	571
kbo-S	146	504	504	491	205	433	96	545
kbo-T	146	518	518	502	205	427	96	544
kbo-r	146	558	558	553	193	418	89	557
kbo-s	146	542	542	537	193	425	89	533
kbo-t	146	549	549	544	193	431	89	557
kbo-tsr	135	496	496	495	164	312	81	521
ll-TtSsRr	157	572	572	572	223	542	97	572
ll-tsrTSR	157	576	576	576	223	547	97	565
syl-l-tsrTSR	95	360	362	331	86	99	65	370
syl-r-tsrTSR	99	389	388	387	80	114	59	395

Ordering	NONE	P-ALL	P-G	P-R	P-R-2	I-ALL	I-R	I-R-P
kbo-R	146	561	561	538	205	421	96	571
kbo-S	146	504	504	491	205	433	96	545
kbo-T	146	518	518	502	205	427	96	544
kbo-r	146	558	558	553	193	418	89	557
kbo-s	146	542	542	537	193	425	89	533
kbo-t	146	549	549	544	193	431	89	557
kbo-tsr	135	496	496	495	164	312	81	521
ll-TtSsRr	157	572	572	572	223	542	97	572
ll-tsrTSR	157	576	576	576	223	547	97	565
syl-l-tsrTSR	95	360	362	331	86	99	96	370
syl-r-tsrTSR	99	389	388	387	80	114	89	395

Table 13: Maximal/Total number of cosets defined - $E_1|E$

C.2 $(2, 5, 7; 2)|E$

Ordering	Precedence on Σ
kbo-A	(a 1) > (b 1) > (A 6) > (B 1)
kbo-B	(a 1) > (b 1) > (A 1) > (B 6)
kbo-a	(a 6) > (b 1) > (A 1) > (B 1)
kbo-b	(a 1) > (b 6) > (A 1) > (B 1)
ll-BbAa	B > b > A > a
syl-l-abAB	a > b > A > B
syl-r-abAB	a > b > A > B

Table 14: Orderings for $(2, 5, 7; 2)|E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	143	171	172	171	149	176	171
kbo-B	163	161	162	165	163	177	165
kbo-a	143	162	163	224	151	157	164
kbo-b	163	237	127	133	163	158	237
ll-BbAa	143	276	276	275	138	176	277
syl-l-abAB	148	164	158	260	148	155	147
syl-r-abAB	148	164	158	247	148	147	163

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	143	177	179	178	152	177	178
kbo-B	163	163	165	167	165	178	168
kbo-a	143	170	172	239	155	165	173
kbo-b	163	246	139	141	165	159	248
ll-BbAa	143	282	287	284	161	177	284
syl-l-abAB	151	175	169	279	152	164	156
syl-r-abAB	151	177	171	272	151	159	172

Table 15: Maximal/Total number of cosets defined - $(2, 5, 7; 2)|E$

C.3 $G^{3,7,17} | < ab, c >$

Ordering	Precedence on Σ
kbo-A	(C 1) > (c 1) > (B 1) > (b 1) > (A 6) > (a 1)
kbo-B	(C 1) > (c 1) > (B 6) > (b 1) > (A 1) > (a 1)
kbo-C	(C 6) > (c 1) > (B 1) > (b 1) > (A 1) > (a 1)
kbo-a	(C 1) > (c 1) > (B 1) > (b 1) > (A 1) > (a 6)
kbo-b	(C 1) > (c 1) > (B 1) > (b 6) > (A 1) > (a 1)
kbo-c	(C 1) > (c 6) > (B 1) > (b 1) > (A 1) > (a 1)
ll-CcBbAa	C > c > B > b > A > a
ll-CcbaBA	C > c > b > a > B > A
syl-l-CcBbAa	C > c > B > b > A > a

Table 16: Orderings for $G^{3,7,17} | < ab, c >$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1153	1640	1875	1198	1153	1153	1640
kbo-B	1125	900	1921	1590	1125	1125	900
kbo-C	1070	1196	1362	2027	1070	1070	1196
kbo-a	1153	821	1778	1001	1153	1153	821
kbo-b	1071	1085	1532	1186	1071	1071	1085
kbo-c	1110	1563	2187	1544	1110	1110	1563
ll-CcBbAa	1153	968	1576	2206	1153	1153	945
ll-CcbaBA	1153	944	1570	2244	1180	1153	945
syl-l-CcBbAa	1004	882	1081	1647	1004	1004	882

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1153	1760	2280	1306	1153	1153	1760
kbo-B	1125	950	2270	1841	1125	1125	950
kbo-C	1075	1293	1489	2193	1075	1075	1293
kbo-a	1153	867	2122	1064	1153	1153	867
kbo-b	1077	1106	1907	1300	1077	1077	1106
kbo-c	1110	1756	2381	1584	1110	1110	1756
ll-CcBbAa	1153	1026	1887	2492	1153	1153	998
ll-CcbaBA	1153	998	1906	2535	1268	1153	999
syl-l-CcBbAa	1040	1012	1198	1853	1040	1040	1012

Table 17: Maximal/Total number of cosets defined - $G^{3,7,17} | < ab, c >$

C.4 $PSL_2(11)|E$

Ordering	Precedence on Σ
kbo-A	(a 1) > (b 1) > (A 6) > (B 1)
kbo-B	(a 1) > (b 1) > (A 1) > (B 6)
kbo-a	(a 6) > (b 1) > (A 1) > (B 1)
kbo-b	(a 1) > (b 6) > (A 1) > (B 1)
ll-BbAa	B > b > A > a
syl-l-abAB	a > b > A > B
syl-r-abAB	a > b > A > B

Table 18: Orderings for $PSL_2(11)|E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	660	660	660	660	660	660	660
kbo-B	660	1006	1005	1013	660	660	1026
kbo-a	660	661	718	660	660	660	661
kbo-b	660	1027	1030	660	660	660	1008
ll-BbAa	660	947	888	897	660	660	967
syl-l-abAB	660	660	660	666	660	660	660
syl-r-abAB	660	660	660	665	660	660	660
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	667	881	826	875	749	703	921
kbo-B	684	1189	1189	1203	753	731	1221
kbo-a	667	772	1146	808	690	698	772
kbo-b	684	1180	1183	758	1092	734	1153
ll-BbAa	684	1082	1027	1036	876	698	1113
syl-l-abAB	671	851	824	760	686	676	851
syl-r-abAB	671	844	823	747	686	676	844

Table 19: Maximal/Total number of cosets defined - $PSL_2(11)|E$

C.5 $(2, 3, 7; 7)|E$

Ordering	Precedence on Σ
kbo-A	(a 1) > (b 1) > (A 6) > (B 1)
kbo-B	(a 1) > (b 1) > (A 1) > (B 6)
kbo-a	(a 6) > (b 1) > (A 1) > (B 1)
kbo-b	(a 1) > (b 6) > (A 1) > (B 1)
ll-BbAa	B > b > A > a
syl-l-abAB	a > b > A > B
syl-r-abAB	a > b > A > B

Table 20: Orderings for $(2, 3, 7; 7)|E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1534	2065	2757	1105	1654	1708	2065
kbo-B	1563	1811	1163	1929	1563	1423	1811
kbo-a	1534	2045	2590	1814	1644	1705	2128
kbo-b	1563	1368	2232	2362	1563	1503	1368
ll-BbAa	1534	1561	1518	2613	1725	1710	1561
syl-l-abAB	1485	2027	1514	2549	1485	1394	2032
syl-r-abAB	1476	2034	1641	2456	1476	1750	2035
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1602	2336	3307	1566	1775	1758	2338
kbo-B	1608	2003	1619	2152	1610	1500	2004
kbo-a	1602	2358	3191	2428	1728	1823	2457
kbo-b	1608	1449	3354	3219	1614	1562	1450
ll-BbAa	1602	1650	1885	2902	1834	1757	1652
syl-l-abAB	1607	2443	2168	3422	1612	1571	2459
syl-r-abAB	1576	2402	2253	3251	1583	1882	2414

Table 21: Maximal/Total number of cosets defined - $(2, 3, 7; 7)|E$

C.6 $M_{11}^{(1)} | < a >$

Ordering	Precedence on Σ
kbo-A	(C 1) > (c 1) > (B 1) > (b 1) > (A 6) > (a 1)
kbo-B	(C 1) > (c 1) > (B 6) > (b 1) > (A 1) > (a 1)
kbo-C	(C 6) > (c 1) > (B 1) > (b 1) > (A 1) > (a 1)
kbo-a	(C 1) > (c 1) > (B 1) > (b 1) > (A 1) > (a 6)
kbo-b	(C 1) > (c 1) > (B 1) > (b 6) > (A 1) > (a 1)
kbo-c	(C 1) > (c 6) > (B 1) > (b 1) > (A 1) > (a 1)
ll-CcBbAa	C > c > B > b > A > a
syl-l-CcBbAa	C > c > B > b > A > a

Table 22: Orderings for $M_{11}^{(1)} | < a >$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	720	720	720	720	720	720	720
kbo-B	720	794	720	720	720	720	810
kbo-C	720	720	720	720	720	720	720
kbo-a	720	720	722	720	720	720	720
kbo-b	720	720	720	720	720	720	720
kbo-c	720	720	740	720	720	720	720
ll-CcBbAa	720	720	720	720	720	720	720
syl-l-CcBbAa	720	720	877	720	720	720	720

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	735	913	883	919	736	733	933
kbo-B	726	1212	1003	1009	888	766	1227
kbo-C	725	1035	883	968	730	741	1045
kbo-a	727	955	845	891	728	748	962
kbo-b	729	1064	918	931	976	733	1139
kbo-c	726	1022	957	906	730	740	997
ll-CcBbAa	720	1069	860	956	999	760	1056
syl-l-CcBbAa	772	1112	1304	1060	772	768	1106

Table 23: Maximal/Total number of cosets defined - $M_{11}^{(1)} | < a >$

C.7 $(8,7|2,3)| < a^2, a^{-1}b >$

Ordering	Precedence on Σ
kbo-A	(a 1) > (b 1) > (A 6) > (B 1)
kbo-B	(a 1) > (b 1) > (A 1) > (B 6)
kbo-a	(a 6) > (b 1) > (A 1) > (B 1)
kbo-b	(a 1) > (b 6) > (A 1) > (B 1)
ll-BbAa	B > b > A > a
syl-l-abAB	a > b > A > B
syl-r-abAB	a > b > A > B

Table 24: Orderings for $(8,7|2,3)| < a^2, a^{-1}b >$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	766	1883	1402	1290	766	1092	1885
kbo-B	1211	1871	1823	1800	1211	1211	1871
kbo-a	901	1839	1634	991	901	901	1839
kbo-b	1211	1804	979	828	1211	1126	1807
ll-BbAa	973	1862	1759	1325	1213	1166	1860
syl-l-abAB	808	1552	907	852	808	837	1552
syl-r-abAB	1096	1566	946	914	1096	946	1566
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	773	1997	1468	1402	773	1094	2000
kbo-B	1213	2006	1956	1932	1213	1213	2006
kbo-a	903	1921	1699	1040	903	903	1921
kbo-b	1213	1890	1022	848	1213	1130	1889
ll-BbAa	975	1936	1834	1389	1304	1167	1933
syl-l-abAB	932	1690	964	945	932	964	1690
syl-r-abAB	1198	1676	965	960	1198	1023	1676

Table 25: Maximal/Total number of cosets defined - $(8,7|2,3)| < a^2, a^{-1}b >$

C.8 $Neu| < a, c >$

Ordering	Precedence on Σ
kbo-A	(a 1) > (c 1) > (c 1) > (A 6) > (B 1) > (C 1)
kbo-B	(a 1) > (b 1) > (c 1) > (A 1) > (B 6) > (C 1)
kbo-C	(a 1) > (b 1) > (c 1) > (A 1) > (B 1) > (C 6)
kbo-a	(a 6) > (b 1) > (c 1) > (A 1) > (B 1) > (C 1)
kbo-b	(a 1) > (b 6) > (c 1) > (A 1) > (B 1) > (C 1)
kbo-c	(a 1) > (b 1) > (c 6) > (A 1) > (B 1) > (C 1)
ll-CcBbAa	C > c > B > b > A > a
syl-l-bBaAcC	b > B > a > A > c > C
syl-r-bBaAcC	b > B > a > A > c > C

Table 26: Orderings for $Neu| < a, c >$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	2356	3607	1701	3524	4404	1873	3596
kbo-B	2053	3491	1955	2700	2053	2308	3537
kbo-C	2134	3492	2326	2611	2134	2910	3498
kbo-a	2473	3354	17874	1917	2061	2682	3353
kbo-b	4107	3491	1640	2559	4107	2650	3488
kbo-c	2254	2740	1775	1917	2254	2455	2808
ll-CcBbAa	1683	3828	2699	3361	9589	1637	3869
syl-l-bBaAcC	1899	2390	1726	1835	1899	1899	2387
syl-r-bBaAcC	2323	6856	5897	4923	2323	2321	6907
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	2369	3717	1795	3633	4473	1898	3705
kbo-B	2075	3607	2061	2819	2077	2413	3667
kbo-C	2151	3627	2441	2720	2157	3068	3631
kbo-a	2488	3458	18426	1988	2080	2744	3457
kbo-b	4119	3616	1738	2652	4119	2680	3612
kbo-c	2264	2815	1860	1994	2268	2517	2889
ll-CcBbAa	1697	3924	2784	3455	10066	1671	3964
syl-l-bBaAcC	2158	2553	1945	2033	2158	2158	2550
syl-r-bBaAcC	2614	7148	6357	5300	2614	2623	7214

Table 27: Maximal/Total number of cosets defined - $Neu| < a, c >$

C.9 $Cam(3)|E$

Ordering	Precedence on Σ
kbo-R	(r 1) > (R 4) > (s 1) > (S 1)
kbo-S	(r 1) > (R 1) > (s 1) > (S 4)
kbo-r	(r 4) > (R 1) > (s 1) > (S 1)
kbo-s	(r 1) > (R 1) > (s 4) > (S 1)
ll-SsRr	S > s > R > r
syl-l-RSrs	R > S > r > s
syl-r-RSrs	R > S > r > s

Table 28: Orderings for $Cam(3)|E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-R	158	1822	1822	1786	1348	345	1916
kbo-S	153	1798	1797	1777	1176	323	1913
kbo-r	158	1820	1820	1730	1380	389	1911
kbo-s	153	1552	1552	1650	1195	374	1146
ll-SsRr	161	1884	1884	1883	1922	386	2005
syl-l-RSrs	131	1365	1357	1353	601	221	1297
syl-r-RSrs	260	1585	1592	1546	267	204	1582
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-R	211	1824	1824	1788	1351	381	1918
kbo-S	207	1801	1800	1780	1191	360	1916
kbo-r	211	1823	1823	1733	1383	424	1914
kbo-s	207	1554	1554	1654	1205	405	1148
ll-SsRr	173	1884	1884	1883	1922	391	2005
syl-l-RSrs	233	1445	1439	1435	685	320	1377
syl-r-RSrs	357	1670	1677	1632	371	295	1667

Table 29: Maximal/Total number of cosets defined - $Cam(3)|E$

C.10 $G^{3,7,16}|E$

Ordering	Precedence on Σ
kbo-A	(C 1) > (c 1) > (B 1) > (b 1) > (A 6) > (a 1)
kbo-B	(C 1) > (c 1) > (B 6) > (b 1) > (A 1) > (a 1)
kbo-C	(C 6) > (c 1) > (B 1) > (b 1) > (A 1) > (a 1)
kbo-a	(C 1) > (c 1) > (B 1) > (b 1) > (A 1) > (a 6)
kbo-b	(C 1) > (c 1) > (B 1) > (b 6) > (A 1) > (a 1)
kbo-c	(C 1) > (c 6) > (B 1) > (b 1) > (A 1) > (a 1)
ll-CcBbAa	C > c > B > b > A > a
ll-CcbaBA	C > c > b > a > B > A
ll-abcABC	a > b > c > A > B > C
syl-l-CcBbAa	C > c > B > b > A > a

Table 30: Orderings for $G^{3,7,16}|E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	74987	61706	61714	68019	65423	54308	61692
kbo-B	71112	62995	63048	43703	59117	51741	62962
kbo-C	70642	61362	61883	47130	72039	71241	61372
kbo-a	74940	56210	56217	54166	62231	54269	56217
kbo-b	71394	65925	65932	57258	57437	68873	65835
kbo-c	70470	53561	54117	63307	61494	56301	53768
ll-CBAcba	75061	58660	58661	74906	54323	62430	59052
ll-CcBbAa	75058	59350	59338	62715	47841	65109	59115
ll-abcABC	75061	57748	57746	60196	53444	58543	57541
syl-l-CcBbAa	60190	43931	64239	50731	60190	60190	43931
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	75174	68546	68554	76584	78496	55399	68595
kbo-B	71112	71157	71262	52149	59845	52429	71115
kbo-C	71227	68565	69089	54757	72785	71913	68579
kbo-a	74941	61436	61453	59945	74987	55158	61453
kbo-b	72263	70489	70493	66932	59350	70983	70395
kbo-c	70470	59937	60579	65942	61783	56888	60179
ll-CBAcba	75462	65147	65148	84627	78545	63529	65694
ll-CcBbAa	75453	65472	65460	70462	71238	66091	65196
ll-abcABC	75462	63154	63152	67241	75677	59510	62918
syl-l-CcBbAa	62131	56621	76038	63959	62133	62131	56621

Table 31: Maximal/Total number of cosets defined - $G^{3,7,16}|E$

C.11 $G(3,3)|E$

Ordering	Precedence on Σ
kbo-A	(a 1) > (b 1) > (A 6) > (B 1)
kbo-B	(a 1) > (b 1) > (A 1) > (B 6)
kbo-a	(a 6) > (b 1) > (A 1) > (B 1)
kbo-b	(a 1) > (b 6) > (A 1) > (B 1)
ll-BbAa	B > b > A > a
ll-bBaA	b > B > a > A
ll-baBA	b > a > B > A
syl-l-BAba	B > A > b > a
syl-l-BbAa	B > b > A > a

Table 32: Orderings for $G(3,3)|E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	7064	64874	64876	54914	17013	7279	66690
kbo-B	7052	68366	68366	55100	19619	7205	66958
kbo-a	17045	70016	70021	67562	16920	13410	69489
kbo-b	16955	68991	68991	66409	19350	5755	68875
ll-BbAa	9753	76022	76022	69897	57708	9253	76276
ll-bBaA	9753	76154	76154	72358	59142	8501	75959
ll-baBA	9753	76039	76039	72036	61084	8468	75830
syl-l-BAab	16698	29187	31186	31567	4917	11576	28930
syl-l-BbAa	9352	37251	35049	28502	7899	8217	33033
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	7095	65034	65036	55056	17068	7345	66853
kbo-B	7084	68550	68550	55263	19668	7276	67142
kbo-a	17276	70193	70198	67745	16976	13513	69666
kbo-b	17203	69171	69171	66588	19409	5777	69050
ll-BbAa	9753	76025	76025	69900	57709	9253	76279
ll-bBaA	9753	76158	76158	72362	59145	8501	75963
ll-baBA	9753	76043	76043	72040	61086	8468	75834
syl-l-BAab	26145	30966	32852	32891	5051	18262	30740
syl-l-BbAa	10984	38747	36549	29470	8108	10450	34366

Table 33: Maximal/Total number of cosets defined - $G(3,3)|E$

D The MacDonald Groups $G(2,m)|E$

D.1 Orderings

Ordering	Precedence on Σ
kbo-A	(a 1) > (b 1) > (A 6) > (B 1)
kbo-B	(a 1) > (b 1) > (A 1) > (B 6)
kbo-a	(a 6) > (b 1) > (A 1) > (B 1)
kbo-b	(a 1) > (b 6) > (A 1) > (B 1)
ll-BbAa	B > b > A > a
syl-l-BAba	B > A > b > a
syl-l-BbAa	B > b > A > a

Table 34: Orderings for $G(2,m)|E$

D.2 $G(2,2)|E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	17	71	71	71	67	47	71
kbo-B	17	71	71	71	67	47	71
kbo-a	17	71	71	71	67	43	71
kbo-b	17	71	71	71	67	43	71
ll-BbAa	17	71	71	71	71	45	71
syl-l-BbAa	17	47	41	49	46	43	52
syl-r-BbAa	17	66	66	61	53	42	66
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	17	71	71	71	67	47	71
kbo-B	17	71	71	71	67	47	71
kbo-a	17	71	71	71	67	43	71
kbo-b	17	71	71	71	67	43	71
ll-BbAa	17	71	71	71	71	45	71
syl-l-BbAa	17	47	41	49	46	43	52
syl-r-BbAa	17	66	66	61	53	42	66

Table 35: Maximal/Total number of cosets defined - $G(2,2)|E$

D.3 $G(2,3)|E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	152	422	422	449	431	137	319
kbo-B	140	458	442	426	593	137	491
kbo-a	153	770	770	496	473	126	362
kbo-b	137	428	395	499	440	110	503
ll-BbAa	161	367	367	380	345	137	371
syl-l-BbAa	132	680	656	409	433	81	528
syl-r-BbAa	78	430	673	548	171	116	582
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	152	422	422	449	431	137	319
kbo-B	142	463	447	430	596	141	496
kbo-a	153	770	770	496	473	126	362
kbo-b	140	435	402	506	445	110	510
ll-BbAa	161	367	367	380	345	137	371
syl-l-BbAa	132	717	693	438	433	108	565
syl-r-BbAa	96	464	708	577	192	118	619

Table 36: Maximal/Total number of cosets defined - $G(2,3)|E$

D.4 $G(2,4)|E$

D.5 $G(2,5)|E$

D.6 $G(2,6)|E$

D.7 $G(2,7)|E$

D.8 $G(2,8)|E$

D.9 $G(2,9)|E$

D.10 $G(2,10)|E$

D.11 $G(2,11)|E$

D.12 $G(2,15)|E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1073	2713	2713	2512	1971	412	2707
kbo-B	1018	2533	2532	1951	958	458	2543
kbo-a	1072	2716	2716	2522	1980	386	2732
kbo-b	1017	2556	2556	2555	1007	701	2591
ll-BbAa	467	2767	2767	2585	2544	424	2770
syl-l-BbAa	627	2142	2111	1498	476	465	2099
syl-r-BbAa	469	1659	1652	1191	490	572	1715
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1074	2713	2713	2512	1971	412	2707
kbo-B	1022	2533	2532	1951	958	460	2543
kbo-a	1075	2716	2716	2522	1980	386	2732
kbo-b	1020	2556	2556	2555	1007	706	2591
ll-BbAa	467	2767	2767	2585	2544	424	2770
syl-l-BbAa	684	2146	2115	1500	476	615	2103
syl-r-BbAa	594	1661	1655	1195	490	598	1722

Table 37: Maximal/Total number of cosets defined - $G(2,4)|E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1158	3664	3664	3136	2732	1231	3690
kbo-B	1050	3441	3441	2287	1039	1315	3452
kbo-a	1147	3686	3686	3180	2817	1159	3686
kbo-b	1076	3465	3465	3447	1131	758	3493
ll-BbAa	1331	3690	3690	3193	3281	1347	3684
syl-l-BbAa	699	2898	2857	1762	1597	521	2869
syl-r-BbAa	710	2095	2068	1298	477	767	2199
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1158	3664	3664	3136	2732	1231	3690
kbo-B	1052	3441	3441	2287	1039	1328	3452
kbo-a	1149	3686	3686	3180	2817	1160	3686
kbo-b	1077	3465	3465	3447	1131	758	3493
ll-BbAa	1331	3690	3690	3193	3281	1347	3684
syl-l-BbAa	738	2904	2863	1766	1807	764	2875
syl-r-BbAa	735	2097	2071	1303	477	784	2209

Table 38: Maximal/Total number of cosets defined - $G(2,5)|E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1199	4739	4739	3924	3715	1448	4762
kbo-B	1129	4467	4459	2666	1218	1601	4497
kbo-a	3156	4724	4724	3931	3718	1359	4736
kbo-b	2921	4557	4555	4515	1360	836	4574
ll-BbAa	1343	4746	4746	3904	4517	1481	4731
syl-l-BbAa	1203	3692	3604	2031	2246	609	3690
syl-r-BbAa	1576	2543	2487	1408	1712	867	2760
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1199	4739	4739	3924	3715	1448	4762
kbo-B	1131	4467	4459	2666	1218	1614	4497
kbo-a	3165	4724	4724	3931	3718	1359	4736
kbo-b	2952	4557	4555	4515	1360	836	4574
ll-BbAa	1343	4746	4746	3904	4517	1481	4731
syl-l-BbAa	1445	3700	3612	2037	2389	891	3698
syl-r-BbAa	1639	2545	2490	1414	1841	890	2775

Table 39: Maximal/Total number of cosets defined - $G(2,6)|E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1225	5930	5930	4637	4703	1598	5952
kbo-B	1212	5626	5610	3095	1513	1907	5659
kbo-a	1225	5911	5911	4664	4728	1506	5935
kbo-b	3215	5749	5746	5668	1613	917	5754
ll-BbAa	1343	5871	5871	4510	5422	1648	5901
syl-l-BbAa	2591	4560	4418	2303	2244	1669	4582
syl-r-BbAa	1616	3015	2928	1511	1713	965	3336
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1225	5930	5930	4637	4703	1598	5952
kbo-B	1212	5626	5610	3095	1514	1922	5659
kbo-a	1225	5911	5911	4664	4728	1513	5935
kbo-b	3224	5749	5746	5668	1614	918	5754
ll-BbAa	1343	5871	5871	4510	5422	1649	5901
syl-l-BbAa	2989	4570	4428	2311	2388	2000	4592
syl-r-BbAa	1675	3017	2931	1518	1843	994	3356

Table 40: Maximal/Total number of cosets defined - $G(2,7)|E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1255	7239	7239	5530	5756	1803	7274
kbo-B	1248	6875	6875	3576	1815	2165	6904
kbo-a	1257	7190	7190	5497	5758	1707	7202
kbo-b	1248	7019	7016	6843	1867	963	7039
ll-BbAa	1343	7117	7117	5289	6758	1758	7168
syl-l-BbAa	2623	5502	5286	2589	2244	1894	5556
syl-r-BbAa	1634	3521	3399	1605	1714	1063	3969
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1255	7239	7239	5530	5756	1803	7274
kbo-B	1248	6875	6875	3576	1817	2179	6904
kbo-a	1257	7190	7190	5497	5758	1714	7202
kbo-b	1248	7019	7016	6843	1868	964	7039
ll-BbAa	1343	7117	7117	5289	6758	1759	7168
syl-l-BbAa	3005	5514	5298	2599	2389	2264	5568
syl-r-BbAa	1689	3523	3402	1613	1845	1098	3994

Table 41: Maximal/Total number of cosets defined - $G(2,8)|E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1266	8645	8645	6350	7047	1955	8662
kbo-B	1264	8199	8199	4151	2117	2389	8239
kbo-a	1266	8662	8662	6323	7020	1836	8685
kbo-b	1267	8394	8388	8174	2209	1027	8412
ll-BbAa	1343	8443	8443	6020	7848	1929	8529
syl-l-BbAa	2624	6518	6218	2889	2244	2119	6612
syl-r-BbAa	1636	4061	3904	1701	1715	1161	4650
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1266	8645	8645	6350	7047	1955	8662
kbo-B	1264	8199	8199	4151	2119	2405	8239
kbo-a	1266	8662	8662	6323	7020	1841	8685
kbo-b	1267	8394	8388	8174	2211	1029	8412
ll-BbAa	1343	8443	8443	6020	7848	1934	8529
syl-l-BbAa	3006	6532	6232	2901	2390	2528	6626
syl-r-BbAa	1690	4063	3907	1710	1847	1202	4680

Table 42: Maximal/Total number of cosets defined - $G(2,9)|E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1275	10142	10142	7369	8099	2178	10156
kbo-B	1277	9647	9647	4617	2339	2607	9715
kbo-a	1275	10202	10202	7305	8117	2032	10182
kbo-b	1278	9854	9848	9558	2482	1080	9867
ll-BbAa	1343	9959	9959	6991	9467	2037	10010
syl-l-BbAa	2624	7608	7214	3203	17342	2344	7750
syl-r-BbAa	1636	4635	4439	1799	1716	1259	5379
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1275	10142	10142	7369	8099	2178	10156
kbo-B	1278	9647	9647	4617	2342	2627	9715
kbo-a	1275	10202	10202	7305	8117	2040	10182
kbo-b	1279	9854	9848	9558	2485	1083	9867
ll-BbAa	1343	9959	9959	6991	9467	2042	10010
syl-l-BbAa	3006	7624	7230	3217	18628	2792	7766
syl-r-BbAa	1691	4637	4442	1809	1849	1306	5414

Table 43: Maximal/Total number of cosets defined - $G(2,10)|E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1279	11830	11833	8239	9677	2278	11818
kbo-B	1280	11114	11114	4983	2582	2806	11180
kbo-a	1279	11818	11818	8096	9644	2133	11792
kbo-b	1281	11410	11403	10949	2642	1092	11433
ll-BbAa	1343	11498	11498	7820	10689	2202	11604
syl-l-BbAa	2624	8772	8274	3531	17973	2569	8970
syl-r-BbAa	1636	5243	5004	1899	1717	1357	6156
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1279	11830	11833	8239	9677	2278	11818
kbo-B	1282	11114	11114	4983	2586	2830	11180
kbo-a	1279	11818	11818	8096	9644	2142	11792
kbo-b	1283	11410	11403	10949	2645	1096	11433
ll-BbAa	1343	11498	11498	7820	10689	2211	11604
syl-l-BbAa	3007	8790	8292	3547	19260	3056	8988
syl-r-BbAa	1692	5245	5007	1910	1851	1410	6196

Table 44: Maximal/Total number of cosets defined - $G(2,11)|E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1285	19459	19459	12476	15979	2875	19489
kbo-B	1326	18410	18410	7141	3604	3682	18540
kbo-a	1285	19336	19336	12197	16000	2675	19387
kbo-b	1326	18931	18925	17614	3644	1263	18955
ll-BbAa	1343	18961	18961	12062	17617	2734	19135
syl-l-BbAa	2624	14168	13154	4983	-	3469	14670
syl-r-BbAa	1721	1749	9744	1636	8015	7564	2319
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1285	19458	19458	12475	15978	2876	19488
kbo-B	1331	18409	18409	7140	3610	3717	18539
kbo-a	1285	19335	19335	12196	15999	2695	19386
kbo-b	1330	18930	18924	17613	3649	1269	18954
ll-BbAa	1343	18960	18960	12061	17616	2754	19134
syl-l-BbAa	3010	14193	13179	5006	-	4111	14695
syl-r-BbAa	1858	1825	9803	1695	8016	7566	2333

Table 45: Maximal/Total number of cosets defined - $G(2,15)|E$

E Examples: Coset Enumeration over General Groups

E.1 $(2, 5, 7; 2)|E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	94	196	196	160	159	136	196
kbo-B	70	132	132	132	104	79	124
kbo-a	94	174	174	152	154	129	178
kbo-b	46	84	84	84	92	78	93
ll-BbAa	94	196	196	160	159	136	196
ll-abAB	94	179	179	154	155	129	179
syl-l-abAB	70	159	153	149	45	83	161
syl-r-abAB	70	137	134	132	114	81	144

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	94	196	196	160	159	136	196
kbo-B	70	132	132	132	104	79	124
kbo-a	94	174	174	152	154	129	178
kbo-b	46	84	84	84	92	78	93
ll-BbAa	94	196	196	160	159	136	196
ll-abAB	94	179	179	154	155	129	179
syl-l-abAB	70	159	153	149	45	83	161
syl-r-abAB	70	145	142	140	116	81	152

Table 46: Orderings for $(2, 5, 7; 2)|E$

E.2 $G^{3,7,17} | < ab, c >$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1153	3936	3618	2308	960	1153	3936
kbo-B	1125	2819	2516	2435	1125	1125	2819
kbo-C	1068	1329	1583	1145	1068	1068	1329
kbo-a	1153	1083	3262	1245	1153	1153	1083
kbo-b	1068	2313	1863	1469	1068	1068	2313
kbo-c	1110	3615	1511	1627	1110	1110	3615
ll-CcBbAa	1153	2493	3187	1297	1154	1153	2493
ll-CcbaBA	1153	2488	1573	1283	1163	1153	2488
syl-l-CcBbAa	1097	3093	1458	913	1097	1097	3093
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1153	4722	4370	2777	1006	1153	4722
kbo-B	1125	3107	3299	3236	1125	1125	3107
kbo-C	1068	1404	1766	1282	1068	1068	1404
kbo-a	1153	1122	4143	1589	1153	1153	1122
kbo-b	1068	2445	2193	1783	1068	1068	2445
kbo-c	1110	3947	1664	1752	1110	1110	3947
ll-CcBbAa	1153	2697	4425	1596	1170	1153	2697
ll-CcbaBA	1153	2699	1908	1584	1200	1153	2699
syl-l-CcBbAa	1124	3569	1788	1045	1124	1124	3569

Table 47: Maximal/Total number of cosets defined - $G^{3,7,17} | < ab, c >$

E.3 $G^{3,7,17}| < ab, c >$ Variant 2

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1153	1125	1819	1501	960	1153	1125
kbo-B	1125	1287	2310	1185	1125	1125	1287
kbo-C	1068	911	1053	1912	1068	1068	911
kbo-a	1153	1060	1001	1656	1153	1153	1060
kbo-b	1068	1565	1605	1665	1068	1068	1565
kbo-c	1110	920	1492	2000	1110	1110	920
ll-CcBbAa	1153	1143	1601	1105	1154	1153	1143
ll-CcbaBA	1153	1089	1954	1105	1163	1153	1089
syl-l-CcBbAa	1097	1348	1591	2148	1097	1097	1348
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1153	1181	2141	1705	1006	1153	1181
kbo-B	1125	1375	2866	1341	1125	1125	1375
kbo-C	1068	995	1140	2048	1068	1068	995
kbo-a	1153	1087	1131	1853	1153	1153	1087
kbo-b	1068	1592	1920	1835	1068	1068	1592
kbo-c	1110	981	1606	2073	1110	1110	981
ll-CcBbAa	1153	1205	1957	1174	1170	1153	1205
ll-CcbaBA	1153	1135	2390	1174	1200	1153	1135
syl-l-CcBbAa	1124	1588	1904	2433	1124	1124	1588

Table 48: Maximal/Total number of cosets defined - $G^{3,7,17}| < ab, c >$

E.4 $PSL_2(11)|E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	660	660	660	660	660	660	660
kbo-B	660	660	660	660	660	660	660
kbo-a	660	660	660	660	660	660	660
kbo-b	660	660	660	660	660	660	660
ll-BbAa	660	660	660	660	660	660	660
syl-l-abAB	660	660	660	660	660	660	660
syl-r-abAB	660	660	660	660	660	660	660
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	660	680	681	679	681	678	686
kbo-B	660	681	681	681	676	666	680
kbo-a	660	721	721	721	680	681	722
kbo-b	660	689	689	689	681	666	689
ll-BbAa	660	689	689	689	681	666	689
syl-l-abAB	660	731	731	731	682	682	732
syl-r-abAB	660	721	721	719	681	682	726

Table 49: Maximal/Total number of cosets defined - $PSL_2(11)|E$

E.5 $(2, 3, 7; 7)|E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1092	2538	2538	2278	1611	1092	2135
kbo-B	1092	1762	1761	1752	1530	1092	1734
kbo-a	1092	2420	2420	2298	1617	1092	2096
kbo-b	1092	1600	1600	1601	1545	1092	1605
ll-BbAa	1092	2538	2538	2278	1611	1092	2135
syl-l-abAB	1092	1837	1837	1846	1653	1092	1747
syl-r-abAB	1092	1664	1664	1647	1393	1092	1646
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1147	2581	2581	2321	1793	1264	2178
kbo-B	1103	1777	1776	1767	1912	1242	1749
kbo-a	1147	2464	2464	2342	1801	1266	2140
kbo-b	1103	1617	1617	1618	1982	1184	1622
ll-BbAa	1147	2581	2581	2321	1793	1264	2178
syl-l-abAB	1104	1850	1850	1859	1864	1201	1760
syl-r-abAB	1104	1673	1673	1656	1604	1197	1655

Table 50: Maximal/Total number of cosets defined - $(2, 3, 7; 7)|E$

E.6 $M_{11}^{(1)} | < a >$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	720	720	720	720	720	720	720
kbo-B	720	1037	756	1059	720	720	1064
kbo-C	720	821	720	720	814	720	791
kbo-a	720	720	720	720	720	720	720
kbo-b	720	1131	720	997	720	720	1156
kbo-c	720	1110	720	832	720	720	1309
ll-CcBbAa	720	776	720	720	720	720	744
syl-l-CcBbAa	720	720	721	720	720	720	780
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	720	966	835	925	727	726	973
kbo-B	720	1324	1257	1473	801	737	1348
kbo-C	754	1191	955	1040	1385	741	1152
kbo-a	722	1109	907	1072	723	727	1031
kbo-b	720	1417	1132	1344	761	733	1435
kbo-c	734	1399	1057	1191	830	802	1596
ll-CcBbAa	720	1149	929	1023	826	731	1113
syl-l-CcBbAa	743	1088	830	1036	759	760	1163

Table 51: Maximal/Total number of cosets defined - $M_{11}^{(1)} | < a >$

E.7 $(8,7|2,3)| < a^2, a^{-1}b >$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	766	1270	776	1088	775	1092	1270
kbo-B	1211	1431	1436	1153	1271	1211	1431
kbo-a	901	554	1124	1212	929	901	554
kbo-b	657	831	1629	1821	657	716	826
ll-BbAa	973	1409	1378	916	957	1166	1409
syl-l-abAB	566	2272	1475	2251	460	448	1405
syl-r-abAB	607	500	505	497	464	551	495
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	773	1375	897	1224	796	1094	1375
kbo-B	1213	1511	1517	1224	1277	1213	1511
kbo-a	903	690	1207	1289	933	903	690
kbo-b	681	914	1751	1925	681	735	921
ll-BbAa	975	1509	1470	991	1110	1167	1509
syl-l-abAB	676	2354	1506	2333	651	549	1499
syl-r-abAB	710	635	627	631	607	652	629

Table 52: Maximal/Total number of cosets defined - $(8,7|2,3)| < a^2, a^{-1}b >$

E.8 $Neu| < a, c >$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1487	3950	2421	3930	3032	1221	3927
kbo-B	1446	7602	5956	7000	5392	1845	7230
kbo-C	1364	4466	2800	4076	640	1045	4604
kbo-a	1489	4984	3055	4868	3559	1563	5161
kbo-b	3103	6791	4843	6462	1869	1972	6775
kbo-c	1249	4407	2570	3835	3798	1451	4852
ll-CcBbAa	1677	3412	2448	3181	5357	3327	3386
syl-l-CcBbAa	560	4682	1691	4548	560	567	4690
syl-r-CcBbAa	920	6571	2836	6429	919	920	6551

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1488	4006	2495	3986	3154	1239	3983
kbo-B	1446	7632	6022	7029	5610	1871	7258
kbo-C	1365	4507	2863	4122	779	1075	4670
kbo-a	1489	4998	3090	4882	3639	1573	5169
kbo-b	3107	6831	4932	6505	2010	1987	6815
kbo-c	1249	4444	2635	3868	3994	1467	4899
ll-CcBbAa	1677	3442	2503	3221	5487	3332	3416
syl-l-CcBbAa	656	4730	1806	4601	656	661	4738
syl-r-CcBbAa	1000	6627	2901	6486	1008	1003	6604

Table 53: Maximal/Total number of cosets defined - $Neu| < a, c >$

E.9 $G^{3,7,16}|E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	74938	0	0	0	0	74938	0
kbo-B	71112	0	0	0	55825	71112	0
kbo-C	70470	0	0	55223	64020	70470	0
kbo-a	74931	0	0	0	0	74931	0
kbo-b	71112	39664	0	0	67592	74889	39635
kbo-c	70470	0	0	53847	57000	73388	0
ll-ABCabc	74940	0	0	0	0	74940	0
ll-CcBbAa	74938	0	0	0	0	77056	0
ll-abcABC	74940	30949	0	0	0	79455	31944
syl-l-CcBbAa	67074	0	58808	61699	59542	67074	0
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	74938	0	0	0	0	74938	0
kbo-B	71111	0	0	0	60693	71111	0
kbo-C	70469	0	0	63592	66389	70469	0
kbo-a	74930	0	0	0	0	74930	0
kbo-b	71111	47384	0	0	69062	75282	47314
kbo-c	70469	0	0	59352	58739	73405	0
ll-CBAcba	74941	0	0	0	0	74941	0
ll-CcBbAa	74937	0	0	0	0	77089	0
ll-abcABC	74941	42538	0	0	0	79511	43388
syl-l-CcBbAa	68594	0	74128	73381	61284	68594	0

Table 54: Maximal/Total number of cosets defined - $G^{3,7,16}|E$

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