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Computer Algebra and Algebraic Geometry – Achievements and Perspectives

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Computer Algebra and Algebraic Geometry — Achievements and Perspectives*

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De computer is niet de steen
maar de slijpsteen der wijzen.
(The computer is not the philosopher's stone
but the philosopher's whetstone.)
Hugo Battus, Rekenen op taal (1989)

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* Extended version of invited talk, delivered at the ISSAC'98 conference at Rostock, 13th – 15th August, 1998.

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1. Preface

In this survey I should like to introduce some concepts of algebraic geometry and try to demonstrate the fruitful interaction between algebraic geometry and computer algebra and, more generally, between mathematics and computer science. One of the aims of this article is to show, by means of examples, the usefulness of computer algebra to mathematical research.

Computer algebra itself is a highly diversified discipline with applications to various areas of mathematics; we find many of these in numerous research papers, in proceedings or in textbooks (cf. Buchberger and Winkler (1998), Cohen, Cuypers and Sterk (1999), Matzat, Greuel and Hiss (1998), ISSAC (1988–1998)). Here we concentrate mainly on Gröbner bases and leave aside many other topics of computer algebra (cf. Davenport, Siret and Tournier (1988), Von zur Gathen and Gerhard (1999), Grabmeier, Kaltofen and Weispfennig (2000)). In particular, we do not mention (multivariate) polynomial factorisation, another major and important tool in computational algebraic geometry. Gröbner bases were introduced originally by Buchberger as a computational tool for testing solvability of a system of polynomial equations, to count the number of solutions (with multiplicities) if this number is finite and, more algebraically, to compute in the quotient ring modulo the given polynomials. Since then, Gröbner bases have become *the* major computational tool, not only in algebraic geometry.

The importance of Gröbner bases for mathematical research in algebraic geometry is obvious and their use needs, nowadays, hardly any justification. Indeed, chapters on Gröbner bases and Buchberger's algorithm (Buchberger, 1965) have been incorporated in many new textbooks on algebraic geometry such as the books of Cox, Little and O'Shea (1992), Cox, Little and O'Shea (1998) or the recent books of Eisenbud (1995), Vasconcelos (1998), not to mention textbooks which are devoted exclusively to Gröbner bases, as Adams and Loustaunou (1994), Becker and Weispfennig (1993), Fröberg (1997).

Computational methods become increasingly important in pure mathematics and the above mentioned books have the effect that Gröbner bases and their applications become a standard part of university courses on algebraic geometry and commutative algebra. One of the reasons is that these methods, together with very efficient computers, allow the treatment of non–trivial examples and, moreover, are applicable to non–mathematical, industrial, technical or economical problems. Another reason is that there is a belief that algorithms can contribute to a deeper understanding of a problem. The human idea of "understanding" is clearly part of the historical, cultural and technical status of the society and nowadays understanding in mathematics requires more and more algorithmic treatment and computational mastering.

On the other hand, it is also obvious that many of the recent deepest achievements in algebraic and arithmetic geometry, such as string theory and mirror symmetry (coming from physics) or Wiles' proof of Fermat's last theorem, just to mention a few, were neither inspired by nor used computer algebra at all. I just mention this in order to stress that no computer algebra system can ever replace, in any significant way, mathematical thinking.

Generally speaking, algorithmic treatment and computational mastering marks not the beginning but the end of a development and already requires an advanced theoretical understanding. In many cases an algorithm is, however, much more than just a careful analysis of known results, it is really a new level of understanding, and an efficient implementation is, in addition, usually a highly nontrivial task. Furthermore, having a computer algebra system which has such algorithms implemented and which is easy to use, it then becomes a powerful tool for the working mathematician, like a calculator for the engineer.

In this connection I should like to stress that having Buchberger's algorithm for computing Gröbner bases of an ideal is, although indispensable, not much more than having +,-,*,/ on a calculator. Nowadays there exist efficient implementations of very involved and sophisticated algorithms (most of them use Gröbner bases in an essential way) allowing the computation of such things as

Hilbert polynomials of graded ideals and modules,
free resolutions of finitely generated modules,
Ext, Tor and cohomology groups,
infinitesimal deformations and obstructions of varieties and singularities,
versal deformations of varieties and singularities,
primary decomposition of ideals,
normalisation of affine rings,
invariant rings of finite and reductive groups,
Puiseux expansion of plane curve singularities,

not to mention the standard operations like ideal and radical membership, ideal intersection, ideal quotient and elimination of variables. All the above–mentioned algorithms are implemented in SINGULAR (Greuel, Pfister and Schönemann, 1990–1998), some of them also in CoCoA (Capani, Niesi and Robbiano, 1995) and Macaulay, resp. Macaulay2 (Bayer and Stillmann, resp. Grayson and Stillmann), to mention computer algebra systems which are designed for use in algebraic geometry and commutative algebra. Even general purpose and commercial systems such as Mathematica, Maple, MuPad etc. offer Gröbner bases and, based on this, libraries treating special problems in algebra and geometry.

It is well–acknowledged that Gröbner bases and Buchberger's algorithm are responsible for the possibility to compute the above objects in affine resp. projective geometry, that is, for non–graded resp. graded ideals and modules over polynomial rings. It is, however, much less known that standard bases ("Gröbner bases" for not necessarily well–orderings) can compute the above objects over the localisation of polynomial rings. This is basically due to Mora's modification of Buchberger's algorithm (Mora (1982)) which has been modified and extended to arbitrary (mixed) monomial orderings in SINGULAR since 1990 and was published in Grassmann et al (1994) and in Greuel and Pfister (1996). We include a brief description in Section 5.

I shall explain how non–well–orderings are intrinsically associated with a ring which may be, for example, a local ring or a tensor product of a local and a polynomial ring. These "mixed rings" are by no means exotic but are necessary for certain algorithms which use tag–variables which have to be eliminated later. The extension of Buchberger's algorithm to non–well–orderings has important applications to problems in local algebraic geometry and singularity theory, such as the computation of

local multiplicities,
Milnor and Tjurina numbers,
syzygies and Hilbert–Samuel functions for local rings

but also to more advanced algorithms such as classification of singularities,

semi–universal deformation of singularities,

computation of moduli spaces,

monodromy of the Gauß–Manin connection.

Moreover, I demonstrate, by means of examples, how some of the above algorithms were used to support mathematical research in a non–trivial manner. These examples belong to the main methods of applying computer algebra successfully:

producing counter examples or giving support to conjectures,

providing evidence and prompting proofs for new theorems,

constructing interesting explicit examples.

The mathematical problems I present were, to a large extent, responsible for the development of SINGULAR, its functionality and speed.

Finally, I point out some open problems in mathematics and non–mathematical applications which are a challenge to computer algebra and where either the knowledge of an algorithm or an efficient implementation is highly desirable.

Acknowledgement: The author was partially supported by the DFG Schwerpunkt "Effiziente Algorithmen für diskrete Probleme". Special thanks to C. Lossen and, in particular, to T. Keilen for preparing the pictures. Finally, I should like to thank the referees for useful comments.

2. Introduction by pictures

The basic problem of algebraic geometry is to understand the set of points $x = (x_1, \ldots, x_n) \in K^n$ satisfying a system of equations

$$\begin{aligned} f_1(x_1, \ldots, x_n) &= 0, \\ &\vdots \\ f_k(x_1, \ldots, x_n) &= 0, \end{aligned}$$

where K is a field and $f_1, \ldots, f_k$ are elements of the polynomial ring $K[x] = K[x_1, \ldots, x_n]$.

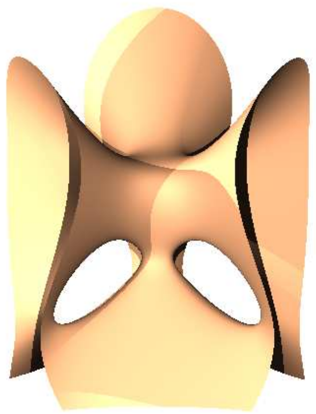
The solution set of $f_1 = 0, \ldots, f_k = 0$ is called the algebraic set, or **algebraic variety** of $f_1, \ldots, f_k$ and is denoted by

$$V = V(f_1, \ldots, f_k).$$

It is easy to see, and important to know, that V depends only on the ideal

$$I = \langle f_1, \ldots, f_k \rangle = \{f \in K[x] \mid f = \sum_{i=1}^{k} a_i f_i,\ a_i \in K[x]\}$$

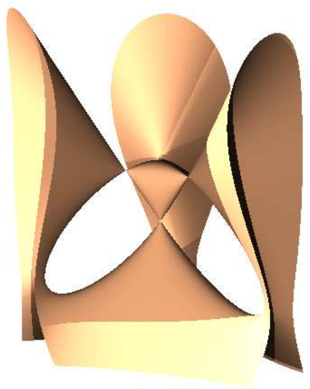
generated by $f_1, \ldots, f_k$ in $K[x]$, that is $V = V(I) = \{x \in K^n \mid f(x) = 0 \,\forall\, f \in I\}$.



The Clebsch Cubic

This is the unique cubic surface which has $\mathbb{S}_5$, the symmetric group of 5 letters, as symmetry group. It is named after its discoverer Alfred Clebsch and has the affine equation

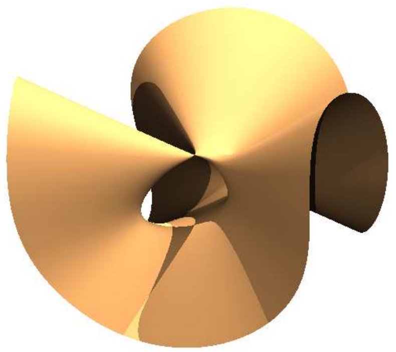
$$\begin{gathered}81(x^3+y^3+z^3)\\ -189(x^2y+x^2z+xy^2+xz^2+y^2z+yz^2)\\ +54xyz+126(xy+xz+yz)\\ -9(x^2+y^2+z^2)-9(x+y+z)+1=0.\end{gathered}$$



The Cayley Cubic

There is a unique cubic surface which has four ordinary double points, usually called the Cayley cubic after its discoverer, Arthur Cayley. It is a degeneration of the Clebsch cubic, has $\mathbb{S}_4$ as symmetry group, and the projective equation is

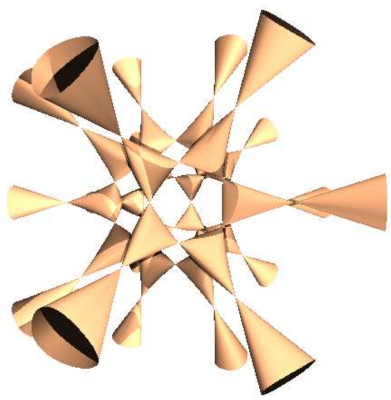
$$z_0z_1z_2+z_0z_1z_3+z_0z_2z_3+z_1z_2z_3=0.$$



A Cubic with a D_4-Singularity

Degenerating the Cayley cubic we receive a D_4-singularity. The affine equation is

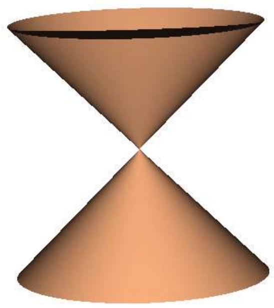
$$x(x^2-y^2)+z^2(1+z)+\tfrac{2}{5}xy+\tfrac{2}{5}yz=0.$$



The Barth Sextic

The equation for this sextic was found by Wolf Barth. It has 65 ordinary double points, the maximal possible number for a sextic. Its affine equation is (with $c=\frac{1+\sqrt{5}}{2}$)

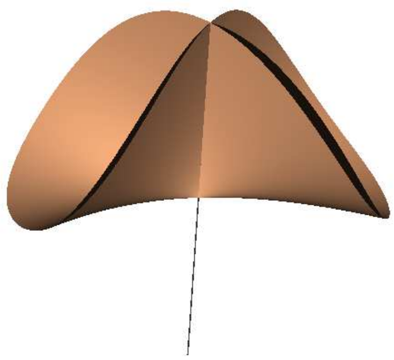
$$\begin{gathered}(8c+4)x^2y^2z^2-c^4(x^4y^2+y^4z^2+x^2z^4)\\ +c^2(x^2y^4+y^2z^4+x^4z^2)\\ -\frac{2c+1}{4}(x^2+y^2+z^2-1)^2=0.\end{gathered}$$



An Ordinary Node

An ordinary node is the most simple singularity. It has the local equation

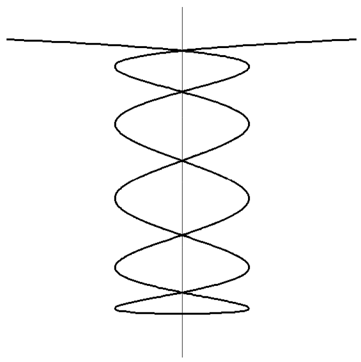
$$x^2+y^2-z^2=0.$$



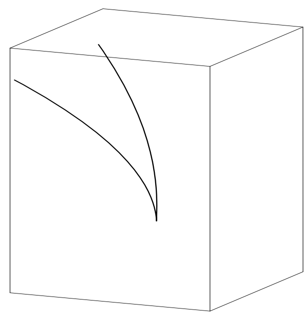
Whitney's Umbrella

The Whitney umbrella is named after Hassler Whitney who studied it in connection with the stratification of analytic spaces. It has the local equation

$$y^2-zx^2=0.$$



A 5-nodal plane curve of degree 11 with equation
$$-16x^2+1048576y^{11}-720896y^9$$
$$+180224y^7-19712y^5+880y^3-11y+\tfrac{1}{2},$$
a deformation of $A_{10}\ :\ y^{11}-x^2=0$.



This space curve is given parametrically by $x=t^4,\ y=t^3,\ z=t^2$, or implicitly by
$$x-z^2=y^2-z^3=0.$$

Of course, if for some polynomial $f \in K[x]$, $f^d|_V = 0$, then $f|_V = 0$ and hence, $V = V(I)$ depends only on the **radical** of I,

$$\sqrt{I} = \{f \in K[x] \mid f^d \in I, \text{ for some } d\}.$$

The biggest ideal determined by V is

$$I(V) = \{f \in K[x] \mid f(x) = 0 \ \forall\, x \in V\},$$

and we have $I \subset \sqrt{I} \subset I(V)$ and $V\big(I(V)\big) = V(\sqrt{I}) = V(I) = V$.

The important **Hilbert Nullstellensatz** states that, for K an algebraically closed field, we have for any variety $V \subset K^n$ and any ideal $J \subset K[x]$,

$$V = V(J) \Rightarrow I(V) = \sqrt{J}$$

(the converse implication being trivial). That is, we can recover the ideal J, up to radical, just from its zero set and, therefore, for fields like $\mathbb{C}$ (but, unfortunately, not for $\mathbb{R}$) geometry and algebra are "almost equal". But almost equal is not equal and we shall have occasion to see that the difference between I and $\sqrt{I}$ has very visible geometric consequences.

Many of the problems in algebra, in particular, computer algebra, have a geometric origin. Therefore, I choose an introduction by means of some pictures of algebraic varieties, some of them being used to illustrate subsequent problems.

The above pictures were not only chosen to illustrate the beauty of algebraic geometric objects but also because these varieties have had some prominent influence on the development of algebraic geometry and singularity theory.

The Clebsch cubic itself has been the object of numerous investigations in global algebraic geometry, the Cayley and the D_4–cubic also, but, moreover, since the D_4–cubic deforms, via the Cayley cubic, to the Clebsch cubic, these first three pictures illustrate deformation theory, an important branch of (computational) algebraic geometry.

The ordinary node, also called A_1–singularity (shown as a surface singularity) is the most simple singularity in any dimension. The Barth sextic illustrates a basic but very difficult and still (in general) unsolved problem: to determine the maximum possible number of singularities on a projective variety of given degree. In Section 7.3 we report on recent progress on this question for plane curves.

Whitney's umbrella was, at the beginning of stratification theory, an important example for the two Whitney conditions. We use the umbrella in Section 4.2 to illustrate that the algebraic concept of normalisation may even lead to a parametrisation of a singular variety, an ultimate goal in many contexts, especially for graphical representations. In general, however, such a parametrisation is not possible, even not locally, if the variety has dimension bigger than one. For curve singularities, on the other hand, the normalisation is always a parametrisation. Indeed, computing the normalisation of the ideal given by the implicit equations for the space curve in the last picture, we obtain the given parametrisation. Conversely, the equations are derived from the parametrisation by eliminating t, where elimination of variables is perhaps the most important basic application of Gröbner bases.

Finally, the 5–nodal plane curve illustrates the global existence problem described in Section 7.2. Moreover, these kind of deformations with the maximal number of nodes play also a prominent role in the local theory of singularities. For instance, from this real picture we can read off the intersection form and, hence, the monodromy of the singularity A_{10} by a beautiful theory of A'Campo and Gusein-Zade. We shall present a completely different, algebraic algorithm to compute the monodromy in Section 6.3.

For more than a hundred years, the connection between algebra and geometry has turned out to be very fruitful and both merged to one of the leading areas in mathematics: algebraic geometry. The relationship between both disciplines can be characterised by saying that algebra provides rigour while geometry provides intuition.

In this connection, I place computer algebra on top of rigour, but I should like to stress its limited value if it is used without intuition.

3. Some problems in algebraic geometry

In this section I shall formulate some of the basic questions and problems arising in algebraic geometry and provide ingredients for certain algorithms. I shall restrict myself to those algorithms where I am somehow familiar with their implementations and which have turned out to be useful in practical applications.

Let me first recall the most basic but also most important applications of Gröbner bases to algebraic constructions (called "Gröbner basics" by Sturmfels). Since these can be found in more or less any textbook dealing with Gröbner bases, I just mention them:

Ideal (resp. module) membership problem
Intersection with subrings (elimination of variables)
Intersection of ideals (resp. submodules)
Zariski closure of the image of a map
Solvability of polynomial equations
Solving polynomial equations
Radical membership
Quotient of ideals
Saturation of ideals
Kernel of a module homomorphism
Kernel of a ring homomorphism
Algebraic relations between polynomials
Hilbert polynomial of graded ideals and modules

The next questions and problems lead to algorithms which are slightly more (some of them much more) involved. They are, nevertheless, still very basic and quite natural. I should like to illustrate them by means of four simple examples, shown in the pictures of this section, referred to as Example 1) – 4):

Assume we are given an ideal $I \subset K[x_1,\dots,x_n]$ by a set of generators $f_1,\dots,f_k \in K[x]$. Consider the following questions and problems:

1 *Is $V(I)$ irreducible or may it be decomposed into several algebraic varieties? If so, find its irreducible components. Algebraically this means to compute a primary decomposition of I or of $\sqrt{I}$, the latter means to compute the associated prime ideals of I.*
Example 1) is irreducible, Example 2) has two components (one of dimension 2 and one of dimension 1), Example 3) has three (one–dimensional) and Example 4) has nine (zero–dimensional) components.

2 *Is I a radical ideal (that is, $I = \sqrt{I}$)? If not, compute its radical $\sqrt{I}$.*
In Examples 1) – 3) I is radical while in Example 4) $\sqrt{I} = \langle y^3 - y, x^3 - x\rangle$, which is much simpler than I. In this example the central point corresponds to $V(\langle x,y\rangle^2)$ which is a **fat point**, that is, it is a solution of I of multiplicity ($= \dim_K K[x,y]/\langle x,y\rangle^2$) bigger than 1 (equal to 3). All other points have multiplicity 1, hence the total number of solutions (counted with multiplicity) is 11. This is a typical example of the kind Buchberger (resp. Gröbner) had in mind at the time of writing his thesis.

3 *A natural question to ask is "how independent are the generators $f_1,\dots,f_k$ of I?", that is, we ask for all relations*

$$(r_1,\dots,r_k) \in K[x]^k, \text{ such that } \sum r_i f_i = 0.$$

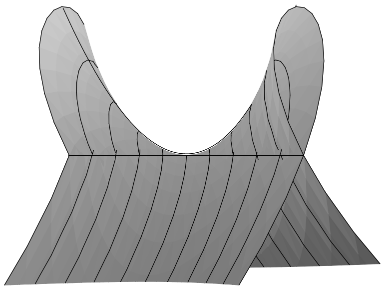
These relations form a submodule of $K[x]^k$, which is called the **syzygy module** of $f_1,\dots,f_k$ and is denoted by $\operatorname{syz}(I)$. It is the kernel of the $K[x]$–linear map

$$K[x]^k \longrightarrow K[x]; \quad (r_1,\dots,r_k) \longmapsto \sum r_i f_i.$$

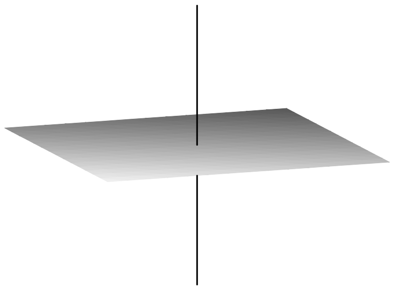
4 *More generally, we may ask for generators of the kernel of a $K[x]$–linear map $K[x]^r \longrightarrow K[x]^s$ or, in other words, for solutions of a system of linear equations over $K[x]$.*
A direct geometric interpretation of syzygies is not so clear, but there are instances where properties of syzygies have important geometric consequences cf. Schreyer (1986).

In Example 1) we have $\operatorname{syz}(I) = 0$, in Example 2), $\operatorname{syz}(I) = \langle(-y, x)\rangle \subset K[x]^2$, in Example 3), $\operatorname{syz}(I) = \langle(-z, y, 0), (-z, 0, x)\rangle \subseteq K[x]^3$ and in Example 4), $\operatorname{syz}(I) \subset K[x]^4$ is generated by $(x, -y, 0, 0), (0, 0, x, -y), (0, x^2 - 1, -y^2 + 1, 0)$.

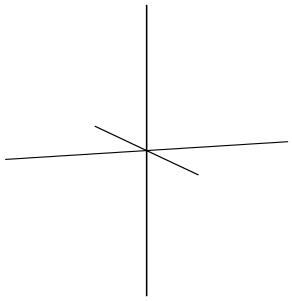
Four examples



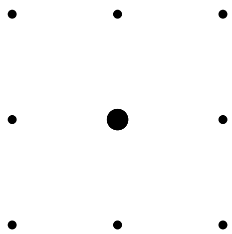
Example 1) the hypersurface
$V(x^2 + y^3 - t^2y^2)$



Example 2) the variety
$V(xz, yz)$



Example 3) the space curve
$V(xy, xz, yz)$



Example 4) the set of points
$V(y^4 - y^2, xy^3 - xy, x^3y - xy, x^4 - x^2)$

5 *A more geometric question is the following. Let $V(I') \subset V(I)$ be a subvariety. How can we describe $V(I) \smallsetminus V(I')$? Algebraically, this amounts to finding generators for the ideal quotient*

$$I : I' = \{f \in K[x] \mid fI' \subset I\}.$$

(The same definition applies if I, I' are submodules of $K[x]^k$.)
Geometrically, $V(I : I')$ is the smallest variety containing $V(I) \smallsetminus V(I')$ which is the (Zariski) closure of $V(I) \smallsetminus V(I')$.
In Example 2) we have $\langle xz, yz\rangle : \langle x, y\rangle = z$ and in Example 3) $\langle xy, xz, yz\rangle : \langle x, y\rangle = \langle z, xy\rangle$, which gives, in both cases, equations for the complement of the z–axis $x = y = 0$. In Example 4) we get $I : \langle x, y\rangle^2 = \langle y(y^2 - 1), x(x^2 - 1), (x^2 - 1)(y^2 - 1)\rangle$ which is the zero set of the eight points $V(I)$ with centre removed.

6 *Geometrically important is the projection of a variety $V(I) \subset K^n$ into a linear subspace K^{n-r}. Given generators $f_1, \ldots, f_k$ of I, we want to find generators for the (closure of the) image of $V(I)$ in $K^{n-r} = \{x | x_1 = \cdots = x_r = 0\}$. The image is defined by the ideal $I \cap K[x_{r+1}, \ldots, x_n]$ and finding generators for this intersection is known as eliminating $x_1, \ldots, x_r$ from $f_1, \ldots, f_k$.*

Projecting the varieties of Examples 1) – 3) to the (x, y)–plane is, in the first two cases, surjective and in the third case it gives the two coordinate axes in the (x, y)–plane. This corresponds to the fact that the intersection with $K[x, y]$ of the first two ideals is 0, while the third one is xy.
Projecting the 9 points of Example 4) to the x–axis we get, by eliminating y, the polynomial $x^2(x-1)(x+1)$, describing the three image points. From a set theoretical point of view this is nice, however it is not satisfactory if we wish to count multiplicities. For example, the two border points are the image of three points each, hence should appear with multiplicity three. That this is not the case can be explained by the fact that elimination computes the annihilator ideal of $K[x, y]/I$ considered as $K[x]$–module (and not the Fitting ideal). This is related to the well–known fact that elimination is not compatible with base change.

7 Another problem is related to the Riemann singularity removable theorem, which states that a function on a complex manifold, which is holomorphic and bounded outside a sub-variety of codimension 1, is actually holomorphic everywhere. This is well–known for open subsets of $\mathbb{C}$, but in higher dimension there exists a second singularity removable theorem, which states that a function, which is holomorphic outside a sub-variety of codimension 2 (no assumption on boundedness), is holomorphic everywhere.
For singular complex varieties this is not true in general, but those for which the two removable theorems hold are called **normal**. Moreover, each reduced variety has a normalisation and there is a morphism with finite fibres from the normalisation to the variety, which is an isomorphism outside the singular locus.
The problem is, given a variety $V(I) \subset K^n$, find a normal variety $V(J) \subset K^m$ and a polynomial map $K^m \longrightarrow K^n$ inducing the normalisation map $V(J) \longrightarrow V(I)$.
The problem can be reduced to irreducible varieties (but need not be, as we shall see) and then the equivalent algebraic problem is to find the normalisation of $K[x_1, \ldots, x_n]/I$, that is the integral closure of $K[x]/I$ in the quotient field of $K[x]/I$ and present this ring as an affine ring $K[y_1, \ldots, y_m]/J$ for some m and J.
For Examples 1) – 4) it can be shown that the normalisation of the first three varieties is smooth, the last two are the disjoint union of the (smooth) components. The corresponding rings are $K[x_1, x_2]$, $K[x_1, x_2] \oplus K[x_3]$, $K[x_1] \oplus K[x_2] \oplus K[x_3]$. The fourth example has no normalisation as it is not reduced.
A related problem is to find, for a non–normal variety V, an ideal H such that $V(H)$ is the non–normal locus of V. The normalisation algorithm described below solves also this problem.
In the examples, the non–normal locus is equal to the singular locus.

8 The significance of **singularities** appears not only in the normalisation problem. The study of singularities is also called **local algebraic geometry** and belongs to the basic tasks of algebraic geometry. Nowadays, singularity theory is a whole subject on its own.
A singularity of a variety is a point which has no neighbourhood in which the Jacobian matrix of the generators has constant rank.
In Example 1) the whole t–axis is singular, in the three other examples only the origin.
One task is to compute generators for the ideal of the singular locus, which is itself a variety. This is just done by computing sub-determinants of the Jacobian matrix, if there are no components of different dimensions. In general, however, we need, additionally, to compute either the equidimensional part and ideal quotients or a primary decomposition.
In Examples 1) – 4), the singular locus is given by $\langle x, y\rangle$, $\langle x, y, z\rangle$, $\langle x, y, z\rangle$, $\langle x, y\rangle^2$, respectively.

9 *Studying a variety $V(I)$, $I = (f_1, \ldots, f_k)$, locally at a singular point, say the origin of K^n, means studying the ideal $IK[x]_{\langle x\rangle}$ generated by I in the local ring*

$$K[x]_{\langle x\rangle} = \left\{ \frac{f}{g} \mid f, g \in K[x], g \notin \langle x_1, \ldots, x_n\rangle \right\}.$$

In this local ring the polynomials g with $g(0) \neq 0$ are units and $K[x]$ is a subring of $K[x]_{\langle x\rangle}$.
Now all the problems we considered above can be formulated for ideals in $K[x]_{\langle x\rangle}$ and modules over $K[x]_{\langle x\rangle}$ instead of $K[x]$.

The geometric problems should be interpreted as properties of the variety in a neighbourhood of the origin, or more generally, the given point.

It should not be surprising that all the above problems have algorithmic and computational solutions, which use, at some place, Gröbner basis methods. Moreover, algorithms for most of these have been implemented quite efficiently in several computer algebra systems, such as CoCoA, cf. Capani, Niesi and Robbiano (1995), Macaulay2, cf. Grayson and Stillmann (1996) and SINGULAR, cf. Greuel, Pfister and Schönemann (1990–1998), the latter also being able to handle, in addition, local questions systematically.

The most complicated problem is the primary decomposition, the latest achievement is the normalisation, both being implemented in SINGULAR.

At first glance, it seems that computation in the localisation $K[x]_{\langle x\rangle}$ requires computation with rational functions. It is an important fact that this is not necessary, but that basically the same algorithms which were developed for $K[x]$ can be used for $K[x]_{\langle x\rangle}$. This is achieved by the choice of a special ordering on the monomials of $K[x]$ where, loosely speaking, the monomials of lower degree are considered to be bigger.

However, such orderings are no longer well–orderings and the classical Buchberger algorithm would not terminate. Mora discovered, cf. Mora (1982), that a different normal form algorithm, or, equivalently, a different division with remainders, leads to termination. Thus, Buchberger's algorithm with Mora's normal form is able to compute in $K[x]_{\langle x\rangle}$ without denominators.

Several algorithms for $K[x]$ use elimination of (some auxiliary extra) variables. But variables to be eliminated have, necessarily, to be well–ordered. Hence, to be able to apply the full power of Gröbner basis methods also for the local ring $K[x]_{\langle x\rangle}$, we need mixed orders, where the monomial ordering restricted to some variables is not a well–ordering, while restricted to other variables it is. In Greuel and Pfister (1996) and Grassmann et al (1994), the authors described a modification of Mora's normal form, which terminates for mixed ordering, and more generally, for any monomial ordering which is compatible with the natural semigroup structure.

4. Some global algorithms

Having mentioned some geometric problems, I shall now illustrate two algorithms related to these problems: primary decomposition and normalisation.

4.1. PRIMARY DECOMPOSITION

Any ideal $I \subset R$ in a Noetherian ring can be written as $I = \bigcap\limits_{i=1}^{r} q_i$ with q_i primary ideals (that is, $q_i \neq R$ and $gf \in q_i$ implies $g \in q_i$ or $f^p \in q_i$ for some $p > 0$).

This generalises the unique factorisation (valid in factorial rings) $f = f_1^{p_1} \cdot \ldots \cdot f_r^{p_r}$ with f_i irreducible, from elements to ideals. In $K[x]$ we have both, unique factorisation and primary decomposition and any algorithm for primary decomposition needs factorisation (because a primary decomposition of a principal ideal $I = \langle f\rangle$ is equivalent to a factorisation of f).

In contrast to factorisation, primary decomposition is, in general, not unique, even if we consider minimal decompositions, that is, the associated primes $p_i = \sqrt{q_i}$ are all distinct and none of the q_i can be omitted in the intersection. However, the minimal (or isolated) primes, that is, the minimal elements of $\mathrm{Ass}\,(I) = \{p_1, \ldots, p_r\}$ with regard to inclusion, are uniquely determined. The minimal primes are the only "geometrically visible" primes in the sense that

$$V(I) = \bigcup_{p_j \in \mathrm{minAss}\,(I)} V(p_j)$$

is the decomposition of $V(I)$ into irreducible components. A non–minimal associated prime $p_i \notin \mathrm{minAss}\,(I)$ is called embedded, because there exists a $p_j \in \mathrm{minAss}\,(I)$, $p_j \subset p_i$. This means geometrically $V(p_i) \subset V(p_j)$, that is, the irreducible component of $V(I)$ corresponding to p_i is embedded in some bigger irreducible component.

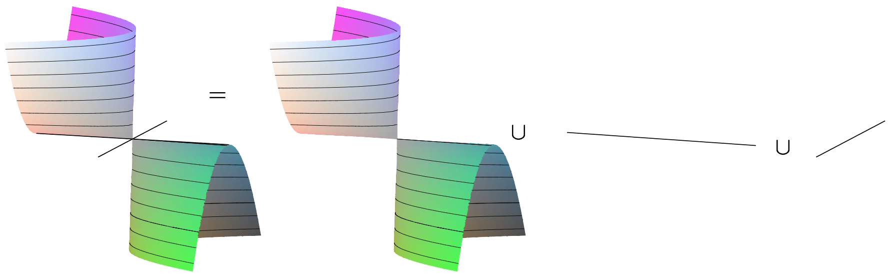
As an example we compute the primary decomposition of the ideal $I = \langle x^2y^3 - x^3yz,\ y^2z - xz^2\rangle$ in SINGULAR, the output being slightly changed in order to save space.

```
LIB "primdec.lib";              //calling library for primary decomposition
ring R  = 0,(x,y,z),dp;
ideal I = x2y3-x3yz,y2z-xz2;
primdecGTZ(I);
==> [1]: [1]:                [2]: [1]:               [3]: [1]:
           _[1]=-y2+xz               _[1]=z2                 _[1]=z
         [2]:                        _[2]=y                  _[2]=x2
           _[1]=-y2+xz          [2]:                    [2]:
                                     _[1]=z                  _[1]=z
                                     _[2]=y                  _[2]=x
```

The result is a list of three pairs of ideals (for each pair, the first ideal is the primary component, the second ideal the corresponding prime component). The second prime component [2] : [2] is embedded in the first [1] : [2]. The first primary component [1] : [1] is already prime, the other two are not.

Hence, $I = (y^2 - xz) \cap (y, z^2) \cap (x^2, z)$ and we obtain:

$$V(I) = \{y^2 - xz = 0\} \cup \underbrace{\{y = z^2 = 0\}}_{\text{(embedded component)}} \cup \{x^2 = z = 0\}$$



Primary decomposition

All known algorithms for primary decompositions in $K[x]$ are quite involved and use many different sub–algorithms from various parts of computer algebra, in particular Gröbner bases, resp. characteristic sets, and multivariate polynomial factorisation over some (algebraic or transcendental) extension of the field K. For an efficient implementation which can treat examples of interest in algebraic geometry, a lot of extra small additional algorithms have to be used. In particular one should use "easy" splitting as soon and as often as possible, see Decker, Greuel and Pfister (1998).

In SINGULAR the algorithms of Gianni, Trager and Zacharias (1988) (which was the first practical and general primary decomposition algorithm), the recent algorithm of Shimoyama and Yokoyama (1996) and some of the homological algebra algorithms for primary decomposition of Eisenbud, Huneke and Vasconcelos (1992) have been implemented. For detailed and improved versions of these algorithms, together with extensive comparisons, see Decker, Greuel and Pfister (1998).

Here are some major ingredients for primary decomposition:

1 Reduction to zero–dimensional primary decomposition (GTZ);
maximal independent sets;
ideal quotient, saturation, intersection.

2 Zero–dimensional primary decomposition (GTZ);
lexicographical Gröbner basis;
factorisation of multivariate polynomials;
generic change of variables;
primitive element computation.

Related algorithms:

1 Computation of the radical;
square–free part of univariate polynomials;
find (random) regular sequences (EHV).
2 Computation of the equidimensional part (EHV);
Ext–annihilators;
ideal quotients, saturation and intersection.

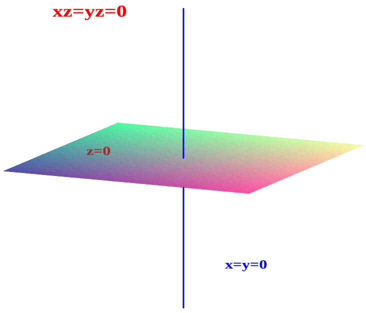
To see how homological algebra comes into play, let us compute the equidimensional part of $V(I)$, that is, the union of all maximal dimensional components of $V(I)$, or, algebraically, the intersection of all minimal primes. Following Eisenbud, Huneke and Vasconcelos (1992), we can calculate the equidimensional part of a variety via Ext–groups:

If $c = \operatorname{codim}_{K[x]}(I)$, then the equidimensional part of I is the annihilator ideal of the module $\operatorname{Ext}^c_{K[x]}(K[x]/I, K[x])$ by Eisenbud, Huneke and Vasconcelos (1992).

For example, the equidimensional part of $V = \{xz = yz = 0\}$ is given by the ideal $\langle z\rangle = \operatorname{ann}\big(\operatorname{Ext}^1(K[x,y,z]/\langle xz, yz\rangle, K[x,y,z])\big)$.

Using SINGULAR, we obtain this via:

```
LIB "homolog.lib";
ring  r  = 0,(x,y,z),dp;
ideal I  = xz, yz;
module M = Ext_R(1,I);
quotient(M,freemodule(nrows(M)));
==> _[1] =z
```



Note that `module M = Ext_R(i,I)` computes a presentation matrix of $\operatorname{Ext}^i(R/I, R)$. Hence, identifying a matrix with its column space in the free module of rank equal to the number of rows, $\operatorname{Ext}^1(R/I,R) = R^n/M$ with $R^n =$ `freemodule(nrows(M))` and, therefore, $\operatorname{Ann}\big(\operatorname{Ext}^1(R/I,R)\big) = M : R^n =$ `quotient(M,freemodule(nrows(M)))`.

Above, we used the procedure `Ext_R(-,-)` from `homolog.lib`. Below we show that the Ext groups can easily be computed directly in a system which offers free resolutions, respectively syzygies, transposition of matrices and presentations of sub-quotients of a free module (`modulo` in SINGULAR). Indeed, the Ext–annihilator can be computed more directly (and faster) without computing the Ext group itself:
Take a free resolution of R/I:

$$0 \longleftarrow R/I \longleftarrow R \longleftarrow R^{n_1} \longleftarrow \cdots .$$

Then consider the dual sequence:

$$0 \longrightarrow \operatorname{Hom}(R,R) \xrightarrow{d^0} \operatorname{Hom}(R^{n_1},R) \xrightarrow{d^1} \cdots .$$

This leads to:
$\operatorname{Ext}^i(R/I,R) = \operatorname{Ker}(d^i)/\operatorname{Im}(d^{i-1})$ and $\operatorname{Ann}\big(\operatorname{Ext}^i(R/I,R)\big) = \operatorname{Im}(d^{i-1}) : \operatorname{Ker}(d^i)$.

The corresponding SINGULAR commands are:

```
int i = 1;
resolution L = res(I,i+1);
module Im    = transpose(L[i]);
module Ker   = syz(transpose(L[i+1]));
module ext   = modulo(Ker,Im);          //the Ext group
ideal ann    = quotient(Im,Ker);        //the Ext-annihilator
```

Since the resolution can be computed by iterated syzygy computation, this is a beautiful example of geometric use of syzygies. However, the algorithm is not at all obvious, but based on the non–trivial theorem of Eisenbud, Huneke and Vasconcelos.

4.2. Normalisation

Another important algorithm is the normalisation of $K[x]/I$ where I is a radical ideal. It can be used as a step in the primary decomposition, as proposed in Eisenbud, Huneke and Vasconcelos (1992), but is also of independent interest. Several algorithms have been proposed, especially by Seidenberg (1975), Stolzenberg (1968), Gianni and Trager (1997), Vasconcelos (1991). It had escaped the computer algebra community, however, that Grauert and Remmert (1971) had given a constructive proof for the ideal of the non–normal locus of a complex space. Within this proof they provide a normality criterion which is essentially an algorithm for computing the normalisation, cf. De Jong (1998). Again, to make the algorithm efficient needed some extra work which is described in Decker, Greuel, De Jong and Pfister (1998). The Grauert–Remmert algorithm is implemented in SINGULAR and seems to be the only full implementation of the normalisation.

Criterion (Grauert and Remmert, 1971): *Let $R = K[x]/I$ with I a radical ideal. Let J be a radical ideal containing a non–zero divisor of R such that $V(J)$ contains the non–normal locus of $V(I)$. Then R is normal if and only if $R = Hom_R(J,J)$.*

For J we may take any ideal so that $V(J)$ contains the singularities of $V(I)$. Since normalisation commutes with localisation, we obtain

Corollary: *$Ann(Hom_R(J,J)/R)$ is an ideal describing the non–normal locus of $V(I)$.*

Now $\mathrm{Hom}_R(J,J)$ is a ring containing R and if $R \subsetneqq \mathrm{Hom}_R(J,J) = R_1$ we can continue with R_1 instead of R and obtain an increasing sequence of rings $R \subset R_1 \subset R_2 \subset \ldots$.

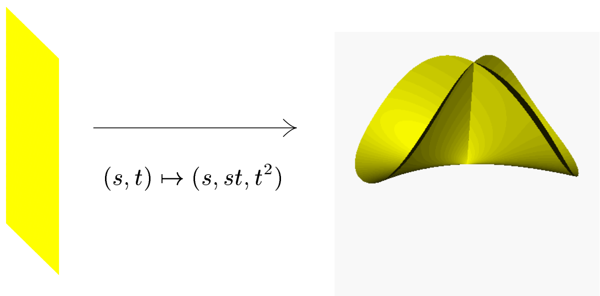
After finitely many steps the sequence becomes stationary (because the normalisation of $R = K[x]/I$ is finite over R) and we reach the normalisation of R by the criterion of Grauert and Remmert.

Ingredients for the normalisation (which is a highly recursive algorithm):

1 Computation of the ideal J of the singular locus of the ideal I;
2 computation of a non–zero divisor for J;
3 ring structure on $\mathrm{Hom}(J,J)$;
4 syzygies, normal forms, ideal quotient.

SINGULAR commands for computation of the normalisation:

```
LIB "normal.lib";
ring S    = 0,(x,y,z),dp;
ideal I   = y2-x2z;
list nor = normal(I);
def R     = nor[1];
setring R;
normap;
==> normap[1]=T(1)
==> normap[2]=T(1)*T(2)
==> normap[3]=T(2)^2
```



In the preceding picture, R, the normalisation of S, is just the polynomial ring in two variables $T(1)$ and $T(2)$. (The "handle" of Whitney's umbrella is invisible in the parametric picture since it requires an imaginary parameter t.)

In several cases the normalisation of a variety is smooth (for example, the normalisation of the discriminant of a versal deformation of an isolated hypersurface singularity) sometimes even an affine space. In this case, the normalisation map provides a parametrisation of the variety. This is the case for the Whitney umbrella: $V = \{y^2 - zx^2 = 0\}$.

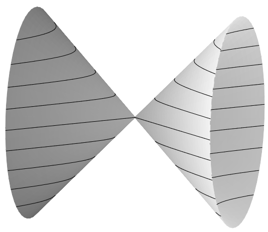
5. Singularities and standard bases

A **(complex) singularity** is, by definition, nothing but a complex analytic germ $(V, 0)$ together with its analytic local ring $R = \mathbb{C}\{x\}/I$, where $\mathbb{C}\{x\}$ is the convergent power series ring in $x = x_1, \ldots, x_n$. For an arbitrary field K let $R = K[[x]]/I$ for some ideal I in the formal power series ring $K[[x]]$. We call $(V, 0) = (\operatorname{Spec} R, \mathfrak{m})$ or just R a singularity ($\mathfrak{m}$ denotes the maximal ideal of the local ring R) and write $K\langle x\rangle$ for the convergent and for the formal power series ring if the statements hold for both.

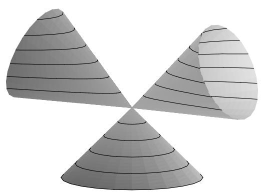
If $I \subset K[x]$ is an ideal with $I \subset \langle x\rangle = \langle x_1, \ldots, x_n\rangle$ then the singularity of $V(I)$ at $0 \in K^n$ is, using the above notation, $K\langle x\rangle/I \cdot K\langle x\rangle$. However, we may also consider the local ring $K[x]_{\langle x\rangle}/I \cdot K[x]_{\langle x\rangle}$ with $K[x]_{\langle x\rangle}$ the localisation of $K[x]$ at $\langle x\rangle$, as the singularity of $V(I)$ at 0. Geometrically, for $K = \mathbb{C}$, the difference is the following: $\mathbb{C}\{x\}/I\mathbb{C}\{x\}$ describes the variety $V(I)$ in an arbitrary small neighbourhood of 0 in the Euclidean topology while $\mathbb{C}[x]_{\langle x\rangle}/I\mathbb{C}[x]_{\langle x\rangle}$ describes $V(I)$ in an arbitrary small neighbourhood of 0 in the (much coarser) Zariski topology.

At the moment, we can compute efficiently only in $K[x]_{\langle x\rangle}$ as we shall explain below. In many cases of interest, we are happy since invariants of $V(I)$ at 0 can be computed in $K[x]_{\langle x\rangle}$ as well as in $K\langle x\rangle$. There are, however, others (such as factorisation), which are completely different in both rings.

Isolated Singularities

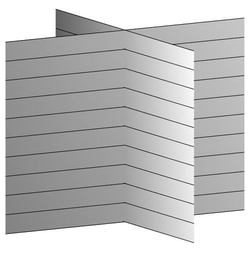


$A_1 : x^2 - y^2 + z^2 = 0$

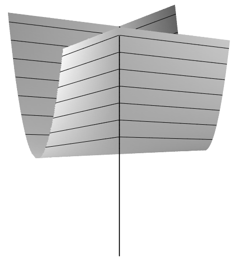


$D_4 : z^3 - zx^2 + y^2 = 0$

Non–isolated singularities



$A_\infty : x^2 - y^2 = 0$



$D_\infty : y^2 - zx^2 = 0$

$(V,0)$ is called **non–singular** or **regular** or **smooth** if $K\langle x\rangle/I$ is isomorphic (as local ring) to a power series ring $K\langle y_1,\dots,y_d\rangle$, or if $K[x]_{\langle x\rangle}/I$ is a regular local ring.

By the implicit function theorem, or by the Jacobian criterion, this is equivalent to the fact that I has a system of generators $g_1,\dots,g_{n-d}$ such that the Jacobian matrix of $g_1,\dots,g_{n-d}$ has rank $n-d$ in some neighbourhood of 0. $(V,0)$ is called an **isolated singularity** if there is a neighbourhood W of 0 such that $W\cap(V\smallsetminus\{0\})$ is regular everywhere.

In order to compute with singularities, we need the notion of standard basis which is a generalisation of the notion of Gröbner basis, cf. Greuel and Pfister (1996), Greuel and Pfister (1998).

A **monomial ordering** is a total order on the set of monomials $\{x^\alpha|\alpha\in\mathbb{N}^n\}$ satisfying

$$x^\alpha > x^\beta \Rightarrow x^{\alpha+\gamma} > x^{\beta+\gamma} \text{ for all } \alpha,\beta,\gamma\in\mathbb{N}^n.$$

We call a monomial ordering $>$ **global** (resp. **local**, resp. **mixed**) if $x_i>1$ for all i (resp. $x_i<1$ for all i, resp. if there exist i,j so that $x_i<1$ and $x_j>1$). This notion is justified by the associated ring to be defined below. Note that $>$ is global if and only if $>$ is a well–ordering (which is usually assumed).

Any $f\in K[x]\smallsetminus\{0\}$ can be written uniquely as $f=cx^\alpha+f'$, with $c\in K\smallsetminus\{0\}$ and $\alpha>\alpha'$ for any non–zero term $c'x^{\alpha'}$ of f'. We set $\mathrm{lm}(f)=x^\alpha$, the **leading monomial** of f and $\mathrm{lc}\,(f)=c$, the **leading coefficient** of f.

For a subset $G\subset K[x]$ we define the **leading ideal** of G as

$$L(G)=\langle\ \mathrm{lm}(g)\mid g\in G\smallsetminus\{0\}\rangle_{K[x]},$$

the ideal generated by the leading monomials in $G\smallsetminus\{0\}$.

So far, the general case is not different to the case of a well–ordering. However, the following definition provides something new for non–global orderings:

For a monomial ordering $>$ define the multiplicatively closed set

$$S_> := \{u\in K[x]\smallsetminus\{0\}\mid \mathrm{lm}\,(u)=1\}$$

and the K–algebra

$$R:=\mathrm{Loc}\,K[x]:=S_>^{-1}K[x]=\{\frac{f}{u}\mid f\in K[x],u\in S_>\},$$

the localisation (ring of fractions) of $K[x]$ with respect to $S_>$. We call $\mathrm{Loc}\,K[x]$ also the **ring associated to** $K[x]$ **and** $>$.

Note that $K[x]\subset\mathrm{Loc}\,K[x]\subset K[x]_{\langle x\rangle}$ and $\mathrm{Loc}\,K[x]=K[x]$ if and only if $>$ is global and $\mathrm{Loc}\,K[x]=K[x]_{\langle x\rangle}$ if and only if $>$ is local (which justifies the names).

Let $>$ be a fixed monomial ordering. In order to have a short notation, I write

$$R:=\mathrm{Loc}\,K[x]=S_>^{-1}K[x]$$

to denote the localisation of $K[x]$ with respect to $>$.

Let $I\subset R$ be an ideal. A finite set $G\subset I$ is called a **standard basis** of I if and only if $L(G)=L(I)$, that is, for any $f\in I\smallsetminus\{0\}$ there exists a $g\in G$ satisfying $\mathrm{lm}\,(g)|\mathrm{lm}\,(f)$.

If the ordering is a well–ordering, then a standard basis G is called a **Gröbner basis**. In this case $R=K[x]$ and, hence, $G\subset I\subset K[x]$.

Standard bases can be computed in the same way as Gröbner bases except that we need a different *normal form*. This was first noticed by Mora (1982) for local orderings (called tangent cone orderings by Mora) and, in general, by Greuel and Pfister (1996), Grassmann et al (1994).

Let $\mathcal{G}$ denote the set of all finite and ordered subsets $G\subset R$. A map

$$\mathrm{NF}\ : R\times\mathcal{G}\to R,\ (f,G)\mapsto\mathrm{NF}\,(f|G),$$

is called a **normal form** on R if, for all f and G,

(i) $\mathrm{NF}\,(f|G)\neq0\Rightarrow\mathrm{lm}\big(\mathrm{NF}\,(f|G)\big)\notin L(G)$,
(ii) $f-\mathrm{NF}\,(f|G)\in\langle G\rangle_R$, the ideal in R generated by G.

NF is called a **weak normal form** if, instead of (ii), only the following condition (ii') holds:

(ii') for each $f \in R$ and each $G \in \mathcal{G}$ there exists a unit $u \in R$, so that $uf - \mathrm{NF}\,(f|G) \in \langle G \rangle_R$.

Moreover, we need (in particular for computing syzygies) (weak) normal forms with **standard representation**: if $G = \{g_1, \ldots, g_k\}$, we can write

$$f - \mathrm{NF}\,(f|G) = \sum_{i=1}^{k} a_i g_i, \quad a_i \in R,$$

such that $\mathrm{lm}\,\big(f - \mathrm{NF}\,(f|G)\big) \geq \mathrm{lm}\,(a_i g_i)$ for all i, that is, no cancellation of bigger leading terms occurs among the $a_i g_i$.

Indeed, if f and G consist of polynomials, we can compute, in finitely many steps, weak normal forms with standard representation such that u and $\mathrm{NF}\,(f|G)$ are polynomials and, hence, compute polynomial standard bases which enjoy most of the properties of Gröbner bases.

Once we have a weak normal form with standard representation, the general standard basis algorithm may be formalised as follows:

STANDARDBASIS(G,NF) [arbitrary monomial ordering]
Input: G a finite and ordered set of polynomials, NF a weak normal form with standard representation.
Output: S a finite set of polynomials which is a standard basis of $\langle G \rangle_R$.

– $S = G$;
– $P = \{(f, g) \mid f, g \in S\}$;
– while $(P \neq \emptyset)$
  choose $(f, g) \in P$;
  $P = P \smallsetminus \{(f, g)\}$;
  $h = \mathrm{NF}\,(\mathrm{spoly}(f, g) \mid S)$;
  if $(h \neq 0)$
    $P = P \cup \{(h, f) \mid f \in S\}$;
    $S = S \cup \{h\}$;
– return S;

Here $\mathrm{spoly}(f, g) = x^{\gamma-\alpha} f - \frac{lc(f)}{lc(g)} x^{\gamma-\beta} g$ denotes the ***s*–polynomial** of f and g where $x^\alpha = \mathrm{lm}\,(f)$, $x^\beta = \mathrm{lm}\,(g)$, $\gamma = lcm(\alpha, \beta)$.

The algorithm terminates by Dickson's lemma or by the noetherian property of the polynomial ring (and since NF terminates). It is correct by Buchberger's criterion, which generalises to non–well–orderings.

If we use Buchberger's normal form below, in the case of a well–ordering, STANDARDBASIS ist just Buchberger's algorithm:

NFBUCHBERGER(f,G) [well–ordering]
Input: G a finite ordered set of polynomials, f a polynomial.
Output: h a normal form of f with respect to G with standard representation.

– $h = f$;
– while ($h \neq 0$ and exist $g \in G$ so that $\mathrm{lm}\,(g) \mid \mathrm{lm}\,(h)$)
  choose any such g;
  $h = \mathrm{spoly}(h, g)$;
– return h;

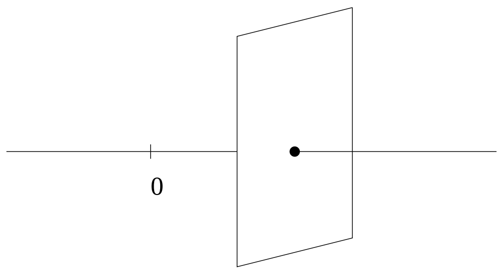
For an algorithm to compute a weak normal form in the case of an arbitrary ordering, we refer to Greuel and Pfister (1996).

To illustrate the difference between local and global orderings, we compute the dimension of a variety at a point and the (global) dimension of the variety.

The **dimension** of the singularity $(V, 0)$, or the dimension of V at 0, is, by definition, the Krull dimension of the analytic local ring $\mathcal{O}_{V,0} = K\langle x \rangle / I$, which is the same as the Krull dimension of the algebraic local ring $K[x]_{\langle x\rangle}/I$ in case $I = \langle f_1, \ldots, f_k\rangle$ is generated by polynomials, which follows easily from the theory of dimensions by Hilbert–Samuel series.

Using this fact, we can compute $\dim(V, 0)$ by computing a standard basis of the ideal $\langle f_1, \ldots, f_k\rangle$ generated in $\operatorname{Loc} K[x]$ with respect to any *local* monomial ordering on $K[x]$. The dimension is equal to the dimension of the corresponding monomial ideal (which is a combinatorial problem).

For example, the dimension of the affine variety $V = V(yx - y, zx - z)$ is 2 but the dimension of the singularity $(V, 0)$ (that is, the dimension of V at the point 0) is 1:



$$V : y(x-1) = z(x-1) = 0,$$
$$\dim(V, 0) = 1,\ \dim V = 2$$

Using SINGULAR we compute first the global dimension with the degree reverse lexicographical ordering denoted by dp and then the local dimension at 0 using the negative degree reverse lexicographical ordering denoted by ds. Note that in the local ring $K[x, y]_{\langle x,y\rangle}$ (represented by the ordering ds) $x - 1$ is a unit.

```
ring R   = 0,(x,y,z),dp;  //global ring
ideal i  = yx-y,zx-z;
ideal si = groebner(i);
si;
==> si[1]=xz-z,           //leading ideal of i is <xz,xy>
==> si[2]=xy-y
dim(si);
==> 2                     //global dimension = dim R/<xz,xy>

ring r   = 0,(x,y,z),ds;  //local ring
ideal i  = yx-y,zx-z;
ideal si = groebner(i);
si;
==> si[1]=y               //leading ideal of i is <y,z>
==> si[2]=z
dim(si);
==> 1                     //local dimension = dim r/<y,z>
```

6. Some local algorithms

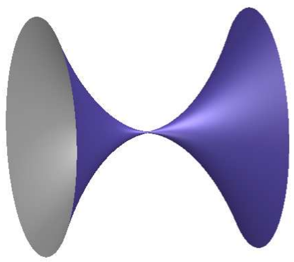
I describe here three algorithms which use, in an essential way, standard bases for local rings: classification of singularities, deformations and the monodromy.

6.1. Classification of singularities

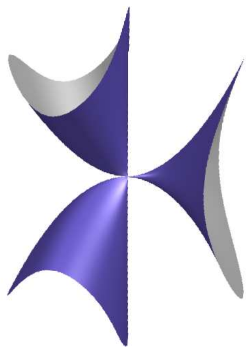
In a tremendous work, V. I. Arnold started, in the late sixties, the classification of hypersurface singularities up to right equivalence. Here f and $g \in K\langle x_1, \ldots, x_n\rangle$ are called right equivalent if they coincide up to analytic coordinate transformation, that is, if there exists a local K–algebra automorphism φ of $K\langle x\rangle$ such that $f = \varphi(g)$. His work culminated in impressive lists of normal forms of singularities and, moreover, in a determinator for singularities which allows the determination of the normal form for a given power series ([AGV, II.16]). This work of Arnold has found numerous applications in various areas of mathematics, including singularity theory, algebraic geometry, differential geometry, differential equations, Lie group theory and theoretical physics. The work of Arnold was continued by C.T.C. Wall and others, cf. Wall (1983), Greuel and Kröning (1990).

Most prominent is the list of ADE or simple or Kleinian singularities, which have appeared in surprisingly different areas of mathematics, and still today, new connections of these singularities to other areas are being discovered (cf. Greuel (1992) for a survey). Here is the list of ADE singularities (the names come from their relation to the simple Lie groups of type A, D and E).

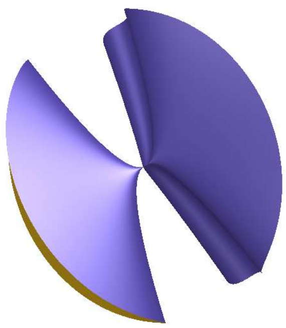
$$\begin{array}{lcll} A_k & : & x_1^{k+1}+x_2^2+x_3^2+\cdots+x_n^2, & k\geq 1\\ D_k & : & x_1(x_1^{k-2}+x_2^2)+x_3^2+\cdots+x_n^2, & k\geq 4\\ E_6 & : & x_1^4+x_2^3+x_3^2+\cdots+x_n^2, & \\ E_7 & : & x_2(x_1^3+x_2^2)+x_3^2+\cdots+x_n^2, & \\ E_8 & : & x_1^5+x_2^3+x_3^2+\cdots+x_n^2. & \end{array}$$



A_3-Singularity



D_6-Singularity



E_7-Singularity

Arnold introduced the concept of "modality", related to Riemann's idea of moduli, into singularity theory and classified all singularities of modality ≤ 2 (and also of Milnor number ≤ 16). The ADE singularities are just the singularities of modality 0. Singularities of modality 1 are the three parabolic singularities:

$$\begin{array}{lcll} \widetilde{E}_6=P_8=T_{333} & : & x^3+y^3+z^3+axyz, & a^3+27\neq 0,\\ \widetilde{E}_7=X_9=T_{244} & : & x^4+y^4+ax^2y^2, & a^2\neq 4,\\ \widetilde{E}_8=J_{10}=T_{236} & : & x^3+y^5+ax^2y^2, & 4a^3+27\neq 0, \end{array}$$

the 3–indexed series of hyperbolic singularities

$$T_{pqr}: x^p+y^q+z^r+axyz, a\neq 0, \frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$$

and 14 exceptional families, cf. Arnold, Gusein-Zade and Varchenko (1985).

The proof of Arnold for his determinator is, to a great part constructive, and has been partly implemented in SINGULAR, cf. Krüger (1997). Although the whole theory and the proofs deal with power series, everything can be reduced to polynomial computation since we deal with isolated singularities, which are **finitely determined**. That is, for an isolated singularity f, there exists an integer k such that f and g are right equivalent if their Taylor expansion coincides up to order k. Therefore, knowing the determinacy k of f, we can replace f by its Taylor polynomial up to order k.

The determinacy can be estimated as the minimal k such that

$$\mathfrak{m}^{k+1}\subset\mathfrak{m}^2\,\mathrm{jacob}(f)$$

where $\mathfrak{m}\subset K\langle x_1,\ldots,x_n\rangle$ is the maximal ideal and $\mathrm{jacob}(f)=\langle\partial f/\partial x_1,\ldots,\partial f/\partial x_n\rangle$. Hence, this k can be computed by computing a standard basis of $\mathfrak{m}^2$ jacob(f) and normal forms of $\mathfrak{m}^i$ with respect to this standard basis for increasing i, using a local monomial ordering. However, there is a much faster way to compute the determinacy directly from a standard basis of $\mathfrak{m}^2$ jacob(f), which is basically the "highest corner" described in Greuel and Pfister (1996).

An important initial step in Arnold's classification is the generalised Morse lemma, or splitting lemma, which says that $f \circ \varphi(x_1, \ldots, x_n) = x_1^2 + \cdots + x_r^2 + g(x_{r+1}, \ldots, x_n)$ for some analytic coordinate change φ and some power series $g \in \mathfrak{m}^3$ if the rank of the Hessian matrix of f at 0 is r.

The determinacy allows the computation of φ up to sufficiently high order and a polynomial g as in the theorem. This has been implemented in SINGULAR and is a cornerstone in classifying hypersurface singularities.

In the following example we use SINGULAR to get the singularity $T_{5,7,11}$ from a database A_L ("Arnold's list"), make some coordinate change and determine then the normal form of the complicated polynomial after coordinate change.

```
LIB "classify.lib";
ring r  = 0,(x,y,z),ds;
poly f  = A_L("T[5,7,11]");
f;
==> xyz+x5+y7+z11
map phi = r, x+z,y-y2,z-x;
poly g   = phi(f);
g;
==> -x2y+yz2+x2y2-y2z2+x5+5x4z+10x3z2+10x2z3+5xz4+z5+y7-7y8+21y9-35y10
==> -x11+35y11+11x10z-55x9z2+165x8z3-330x7z4+462x6z5-462x5z6+330x4z7
==> -165x3z8+55x2z9-11xz10+z11-21y12+7y13-y14
classify(g);
==> The singularity ... is R-equivalent to T[p,q,r]=T[5,7,11]
```

Ingredients for the classification of singularities:

1 standard bases for local and global orderings;
2 computation of invariants (Milnor number, determinacy, ...);
3 generalised Morse lemma;
4 syzygies for local orderings.

Beyond classification by normal forms, the construction of moduli spaces for singularities, for varieties or for vector bundles is a pretentious goal, theoretically as well as computational. First steps towards this goal for singularities have been undertaken in T. Bayer (2000) and Frühbis-Krüger (2000).

6.2. DEFORMATIONS

Consider a singularity $(V, 0)$ given by power series $f_1, \ldots, f_k \in K\langle x_1, \ldots, x_n\rangle$. The idea of deformation theory is to perturb the defining functions, that is to consider power series $F_1(t, x), \ldots, F_k(t, x)$ with $F_i(0, x) = f_i(x)$, where $t \in S$ may be considered as a small parameter of a parameter space S (containing 0).

For $t \in S$ the power series $f_{i,t}(x) = F_i(t, x)$ define a singularity V_t, which is a perturbation of $V = V_0$ for $t \neq 0$ close to 0. It may be hoped that V_t is simpler than V_0 but still contains enough information about V_0. For this hope to be fulfilled, it is, however, necessary to restrict the possible perturbations of the equations to *flat* perturbations, which are called deformations.

Grothendieck's criterion of flatness states that the perturbation given by the F_i is **flat** if and only if any relation between the f_i, say

$$\sum r_i(x) f_i(x) = 0,$$

lifts to a relation

$$\sum R_i(t,x)F_i(t,x) = 0,$$

with $R_i(x,0) = r_i(x)$. Equivalently, for any generator $(r_1,\dots,r_k)$ of $\operatorname{syz}(f_1,\dots,f_k)$ there exists an element $(R_1,\dots,R_k) \in \operatorname{syz}(F_1,\dots,F_k)$ satisfying $R_i(0,x) = r_i(x)$. Hence, syzygies with respect to local orderings come into play.

There exists the notion of a semi–universal deformation of $(V,0)$ which contains essentially all information about all deformations of $(V,0)$.

For an isolated hypersurface singularity $f(x_1,\dots,x_n)$ the semi–universal deformation is given by

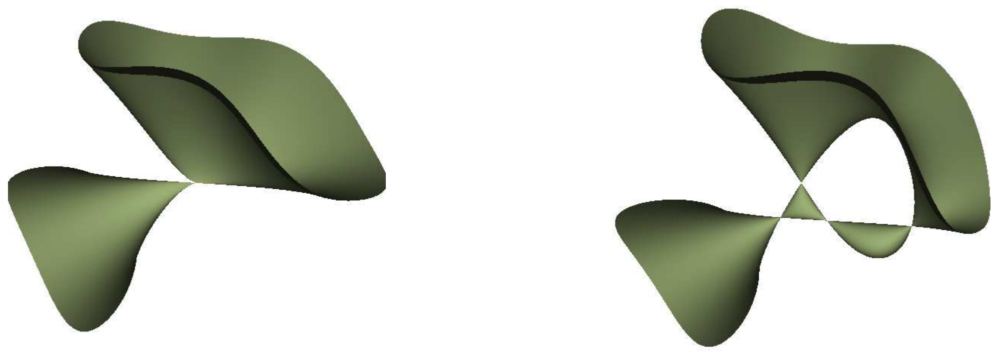
$$F(t,x) = f(x) + \sum_{j=1}^{\tau} t_j g_j(x),$$

where $1 =: g_1, g_2, \dots, g_\tau$ represent a K–basis of the Tjurina algebra

$$K\langle x\rangle / \langle f, \partial f/\partial x_1, \dots, \partial f/\partial x_n\rangle,$$

$\tau = \dim_K K\langle x\rangle/\langle f, \partial f/\partial x_1,\dots,\partial f/\partial x_n\rangle$ being the Tjurina number.

To compute $g_1,\dots,g_\tau$ we only need to compute a standard basis of the ideal $\langle f, \frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}\rangle$ with respect to a local ordering and then compute a basis of $K[x]$ modulo the leading monomials of the standard basis. For complete intersections we have similar formulas.



Deformation of E_7 in $4A_1$

For non–hypersurface singularities, the semi–universal deformation is much more complicated and up to now no finite algorithm is known in general. However, there exists an algorithm to compute this deformation up to arbitrary high order cf. Laudal (1979), Martin (1998), which is implemented in SINGULAR.

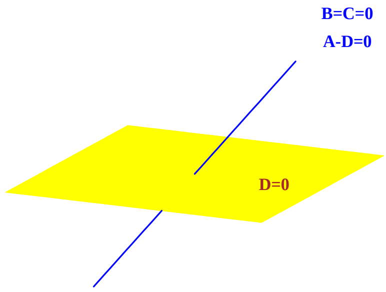
As an example we calculate the base space of the semi–universal deformation of the normal surface singularity, being the cone over the rational normal curve C of degree 4, parametrised by $t \mapsto (t,t^2,t^3,t^4)$.

Homogeneous equations for the cone over C are given by the 2×2–minors of the matrix:

$$m = \begin{pmatrix} x & y & z & u \\ y & z & u & v \end{pmatrix} \in \operatorname{Mat}_{2\times 4}(K[x,y,z,u,v]).$$

SINGULAR commands for computing the semi–universal deformation:

```
LIB "deform.lib";
ring r         = 0,(x,y,z,u,v),ds;
matrix m[2][4] = x,y,z,u,y,z,u,v;
ideal f        = minor(m,2);
versal(f);
setring Px;
Fs;
==> Fs[1,1]=-u2+zv+Bu+Dv
==> Fs[1,2]=-zu+yv-Au+Du
==> Fs[1,3]=-yu+xv+Cu+Dz
==> Fs[1,4]=z2-yu+Az+By
==> Fs[1,5]=yz-xu+Bx-Cz
==> Fs[1,6]=-y2+xz+Ax+Cy
Js;
==> Js[1,1]=BD
==> Js[1,2]=AD-D2
==> Js[1,3]=-CD
```



The ideal $Js = \langle BD, AD - D^2, -CD\rangle \subset K[A,B,C,D]$ defines the required base space which consists of a 3–dimensional component ($D = 0$) and a transversal 1–dimensional component ($B = C = A - D = 0$). This was the first example, found by Pinkham, of a base space of a normal surface having several components of different dimensions.

The full versal deformation is given by the map (`Fs` and `Js` as above)

$$K[[A,B,C,D]]/J_s \longrightarrow K[[A,B,C,D,x,y,z,u,v]]/J_s + F_s.$$

Although, in general, the equations for the versal deformation are formal power series, in many cases of interest (as in the example above) the algorithm terminates and the resulting ideals are polynomial.

Ingredients for the semi–universal deformation algorithm:

1 Computation of standard bases, normal forms and resolutions for local orderings;
2 computation of Ext groups (cf. 4.1) for computing infinitesimal deformations and obstructions;
3 computation of Massey products for determining obstructions to lift, recursively, infinitesimal deformations of a given order to higher order;
4 one of the main difficulties in point 3 is the necessity to compute a completely reduced normal form with respect to a local ordering. In general, such a normal form exists only as formal power series. In the present situation, however, the reduction has to be carried out only for a subset of the variables in a fixed degree and, hence, the complete reduction is finite.

6.3. THE MONODROMY

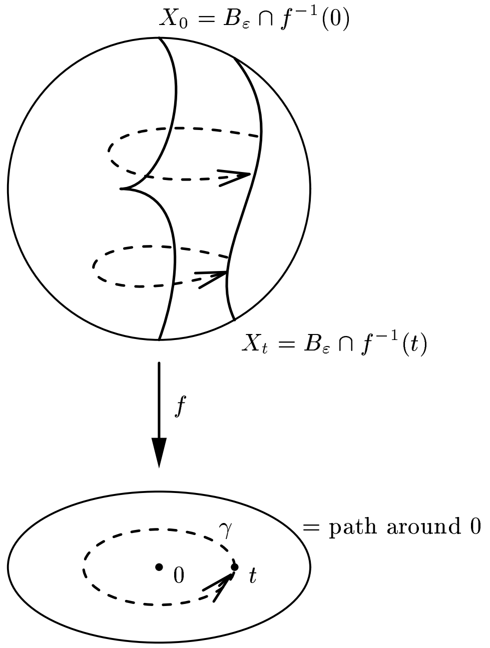
Let $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ be a convergent power series (in practice a polynomial) with isolated singularity at 0 and $\mu = \dim_{\mathbb{C}} \mathbb{C}\{x\}/\langle f_{x_1}, \ldots, f_{x_n}\rangle$ the Milnor number of f.

Then f defines in an ε-ball B_ε around 0 a holomorphic function to $\mathbb{C}$, $f : B_\varepsilon \longrightarrow \mathbb{C}$.

The simple, counterclockwise path γ in $\mathbb{C}$ around 0 induces a C^∞–diffeomorphism of X_t ($t \neq 0$) (as indicated in the figure) and an automorphism of the singular cohomology group $H^n(X_t, \mathbb{C})$ which is, by a theorem of Milnor, a μ–dimensional $\mathbb{C}$–vector space. This automorphism

$$T : H^n(X_t, \mathbb{C}) \xrightarrow{\cong} H^n(X_t, \mathbb{C})$$

is called the local **Picard–Lefschetz monodromy** of f. We address the problem of computing the Jordan normal form of T.



The first important theorem is the
Monodromy theorem (Deligne 1970, Brieskorn 1971): *The eigenvalues of T are roots of unity, that is, we have*

$$T = e^{2\pi i M},$$

where M is a complex matrix with eigenvalues in $\mathbb{Q}$.

Hence, we are left with the problem of computing the Jordan normal form of M.

It is not at all clear that the purely topological definition of T allows an algebraic and computable interpretation. The first hint in this direction is that we can compute $\dim_{\mathbb{C}} H^n(X_t, \mathbb{C})$, according to Milnor's theorem, algebraically by the formula for μ given above.

Since X_t is a complex Stein manifold, its complex cohomology can be computed, via the holomorphic de Rham theorem, with holomorphic differential forms, which is the starting point for computing the monodromy.

To cut a long story short, we just mention, cf. Brieskorn (1970), Greuel (1975) that

$$H' = \Omega^n / df \wedge \Omega^{n-1} + d\Omega^{n-1},$$
$$H'' = \Omega^{n+1} / df \wedge d\Omega^{n-1}$$

are free $\mathbb{C}\{t\}$–modules (via $f^* : \mathbb{C}\{t\} \longrightarrow \mathbb{C}\{f\} \subset \mathbb{C}\{x\}$) of rank equal to μ. Here $(\Omega^\bullet, d)$ denotes the complex of holomorphic differential forms in $(\mathbb{C}^n, 0)$. H' and H'' are called Brieskorn lattices.

We define the local **Gauß–Manin connection** of f as

$$\bigtriangledown : df \wedge H' = df \wedge \Omega^n / df \wedge d\Omega^{n-1} \longrightarrow H'',$$
$$\bigtriangledown [df \wedge \omega] = [d\omega].$$

Note that $\bigtriangledown(df \wedge H') \not\subset df \wedge H'$, that is, $\bigtriangledown$ has a pole at 0. Tensoring with $\mathbb{C}(t)$, the quotient field of $\mathbb{C}\{t\}$, we can extend $\bigtriangledown$ to a meromorphic connection

$$\bigtriangledown : H'' \underset{\mathbb{C}\{t\}}{\otimes} \mathbb{C}(t) \longrightarrow H'' \underset{\mathbb{C}\{t\}}{\otimes} \mathbb{C}(t)$$

(since df $\wedge H' \otimes \mathbb{C}(t) = H'' \otimes \mathbb{C}(t)$) using the Leibniz rule $\bigtriangledown(\omega y) = \bigtriangledown(\omega) y + \omega dy/dt$.

With respect to a basis $\omega_1, \ldots, \omega_\mu$ of H'' we have $\bigtriangledown(\omega_i) = \sum_j a_{ji}\omega_j$ and, for any $\omega = \sum_i \omega_i y_i$,

$\nabla(\omega) = \sum\limits_{i,j} a_{ji} y_i + \sum\limits_i \omega_i dy_i/dt$. Hence, the kernel of ∇, together with a basis of H'', is the same as the solutions of the system of rank μ of ordinary differential equations

$$\frac{dy}{dt} = -Ay, \qquad A = (a_{ij}) \in \operatorname{Mat}\bigl(\mu \times \mu, \mathbb{C}(t)\bigr)$$

in a neighbourhood of 0 in $\mathbb{C}$. The connection matrix, $A = \sum\limits_{i \geq -p} A_i t^i$, $A_i \in \operatorname{Mat}(\mu \times \mu, \mathbb{C})$, has a pole at $t = 0$ and is holomorphic for $t \neq 0$. If $\phi_t = (\phi_1, \ldots, \phi_\mu)$ is a fundamental system of solutions at a point $t \neq 0$, then the analytic continuation of ϕ_t along the path γ transforms ϕ_t into another fundamental system ϕ'_t which satisfies $\phi'_t = T_\nabla \phi_t$ for some matrix $T_\nabla \in \operatorname{GL}(\mu, \mathbb{C})$.

Fundamental fact (Brieskorn, 1971): *The Picard–Lefschetz monodromy T coincides with the monodromy T_∇ of the Gauß–Manin connection.*

Brieskorn used this fact to describe in Brieskorn (1970) the essential steps for an algorithm to compute the characteristic polynomial of T. Results of Gerard and Levelt allowed the extension of this algorithm to compute the Jordan normal form of T, cf. Gerard and Levelt (1973). An early implementation by Nacken in Maple was not very efficient. Recently, Schulze (1999) implemented an improved version in SINGULAR which is able to compute interesting examples.

The algorithm uses another basic theorem, the

Regularity Theorem (Brieskorn, 1971): *The Gauß–Manin connection has a regular singular point at 0, that is, there exists a basis of some lattice in $H'' \otimes \mathbb{C}(t)$ such that the connection matrix A has pole of order 1.*

Basically, if $A = A_{-1}t^{-1} + A_0 + A_1 t + \cdots$ has a simple pole, then $T = e^{2\pi i} A_{-1}$ is the monodromy (this holds if the eigenvalues of A_{-1} do not differ by integers which can be achieved algorithmically).

SINGULAR example:

```
LIB "mondromy.lib";
ring R   = 0,(x,y),ds;
poly f   = x2y2+x6+y6;   //example of A'Campo (monodromy is not diagonalisable)
matrix M = monodromy(f);
print(jordanform(M));
==> 1/2,1,  0,  0,  0,  0,  0,0,0,0,  0,  0,  0,
==> 0,  1/2,0,  0,  0,  0,  0,0,0,0,  0,  0,  0,
==> 0,  0,  2/3,0,  0,  0,  0,0,0,0,  0,  0,  0,
==> 0,  0,  0,  2/3,0,  0,  0,0,0,0,  0,  0,  0,
==> 0,  0,  0,  0,  5/6,0,  0,0,0,0,  0,  0,  0,
==> 0,  0,  0,  0,  0,  5/6,0,0,0,0,  0,  0,  0,
==> 0,  0,  0,  0,  0,  0,  1,0,0,0,  0,  0,  0,
==> 0,  0,  0,  0,  0,  0,  0,1,0,0,  0,  0,  0,
==> 0,  0,  0,  0,  0,  0,  0,0,1,0,  0,  0,  0,
==> 0,  0,  0,  0,  0,  0,  0,0,0,7/6,0,  0,  0,
==> 0,  0,  0,  0,  0,  0,  0,0,0,0,  7/6,0,  0,
==> 0,  0,  0,  0,  0,  0,  0,0,0,0,  0,  4/3,0,
==> 0,  0,  0,  0,  0,  0,  0,0,0,0,  0,  0,  4/3
```

Ingredients for the monodromy algorithm:

1 Computation of standard bases and normal forms for local orderings;
2 computation of Milnor number;
3 Taylor expansion of units in $K[x]_{\langle x \rangle}$ up to sufficiently high order;
4 computation of the connection matrix on increasing lattices in $H'' \otimes \mathbb{C}(t)$ up to sufficiently high order (until saturation) by linear algebra over $\mathbb{Q}$;

5 computation of the transformation matrix to a simple pole by linear algebra over $\mathbb{Q}$;
6 factorisation of univariate polynomials (for Jordan normal form).

The most expensive parts are certain normal form computations for a local ordering and the linear algebra part because here one has to deal iteratively with matrices with several thousand rows and columns. It turned out that the SINGULAR implementation of modules (considered as sparse matrices) and the Buchberger inter-reduction is sufficiently efficient (though not the best possible) for such tasks.

7. Computer algebra solutions to singularity problems

We present three examples which demonstrate, in a somewhat typical way, the use of computer algebra as stated in the preface:

1 producing counter examples;
2 providing evidence and prompting proofs for new theorems;
3 constructing interesting explicit examples;

7.1. EXACTNESS OF THE POINCAREÉ COMPLEX

The first application is a counterexample to a conjectured generalisation of a theorem of Saito (1971) which says that, for an isolated hypersurface singularity, the exactness of the Poincaré complex implies that the defining polynomial is, after some analytic coordinate change, weighted homogeneous.

Theorem (Saito, 1971):
If $f : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ has an isolated singularity at 0, then the following are equivalent:

1 *$X = f^{-1}(0)$ is weighted homogeneous for a suitable choice of coordinates.*
2 *$\mu = \tau$ where $\mu = \dim_{\mathbb{C}} \mathbb{C}\{x\}/\left(\frac{\partial f}{\partial x_i}\right)$ is the Milnor number and*
$\tau = \dim_{\mathbb{C}} \mathbb{C}\{x\}/\left(f, \frac{\partial f}{\partial x_i}\right)$ the Tjurina number.
3 *The holomorphic Poincaré complex*

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \longrightarrow \ldots \longrightarrow \Omega_X^n \longrightarrow 0$$

is exact.

A natural problem is whether the theorem holds also for complete intersections $X = f^{-1}(0)$ with $f = (f_1, \ldots, f_k) : \mathbb{C}^{n+k} \longrightarrow \mathbb{C}^k$. Again we have a Milnor number μ and a Tjurina number τ,

$$\mu = \sum_{i=1}^{k} (-1)^{i-1} \dim_{\mathbb{C}} \mathbb{C}\{x\}/\left(f_1, \ldots, f_{i-1}, \left|\frac{\partial(f_1, \ldots, f_i)}{\partial(x_{j_1}, \ldots, x_{j_i})}\right|\right)$$
$$\tau = \dim_{\mathbb{C}} \mathbb{C}\{x\}^k/\big((f_1, \ldots, f_k)\mathbb{C}\{\underline{x}\}^k + Df(\mathbb{C}\{\underline{x}\}^{n+k})\big).$$

Theoretical reduction (Greuel, Martin and Pfister, 1985):
If X is a complete intersection of dimension 1, then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3).
If $k = 2$, then (3) $\Rightarrow$ (2) if $\mu = \dim_{\mathbb{C}} \Omega_X^2 - \dim_{\mathbb{C}} \Omega_X^3$ and if f_1, f_2 are weighted homogeneous.

Pfister and Schönemann (1989) showed that (3) $\Rightarrow$ (2) does **not** hold in general:

$$f_1 = xy + z^{\ell-1}, \quad f_2 = xz + y^{k-1} + yz^2 \quad (4 \le \ell \le k,\ k \ge 5)$$

is a counterexample.

The proof uses an implementation of the standard basis algorithm in a forerunner of SINGULAR and goes as follows:

1 Compute $\mu, \dim_{\mathbb{C}} \Omega_X^2, \dim_{\mathbb{C}} \Omega_X^3$ to show that $\Omega_X^\bullet$ is exact;
2 compute τ.
One obtains $\mu = \tau + 1$, that is, X is not weighted homogeneous.

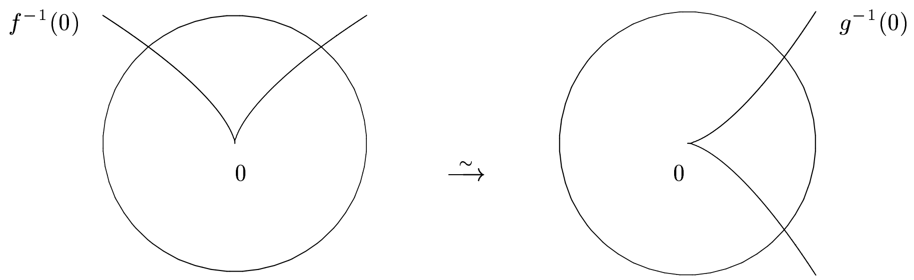
To do this we must be able to compute standard bases of modules over local rings.
The counterexample was found through a computer search in a list of singularities classified by Wall (1983).

7.2. Zariski's multiplicity conjecture

The attempt to find a counterexample to Zariski's multiplicity conjecture — which says that the multiplicity (lowest degree) of a power series is an invariant of the embedded topological type — led, finally after many experiments and computations, to a partial proof of this conjecture. For this, an extremely fast standard basis computation for 0–dimensional ideals in a local ring was necessary.
The following question was posed by Zariski (1971) in his retiring address to the AMS in 1971.
Let $f = \sum c_\alpha x^\alpha \in \mathbb{C}\{x_1, \ldots, x_n\}$, $f(0) = 0$, be a hypersurface singularity, and let $\mathrm{mult}\,(f) := \min\{|\alpha| \bigm| c_\alpha \neq 0\}$ be the multiplicity.

We say that f and g are topological equivalent, $f \overset{\text{top}}{\sim} g$, if there is a homeomorphism

$$(B, f^{-1}(0) \cap B, 0) \xrightarrow{\sim} (B, g^{-1}(0) \cap B, 0)$$



Zariski's conjecture may be stated as: $f \overset{\text{top}}{\sim} g \Rightarrow \mathrm{mult}\,(f) = \mathrm{mult}\,(g)$.

The result is known to be true for curves (Zariski, Lê) and weighted homogeneous singularities Greuel (1986), O'Shea (1987).
Our attempt to find a counterexample was as follows:
Consider deformations of $f = f_0$:

$$f_t(x) = f(x) + tg(x,t), \quad |t| \text{ small.}$$

Then use the theoretical fact proved by Lê and Ramanujam:

$$f_0 \overset{\text{top}}{\sim} f_t \Rightarrow \mu(f_0) = \mu(f_t)$$

("$\Leftarrow$" holds also, except for $n = 3$, where the answer is still unknown) where $\mu(f_0)$ respectively $\mu(f_t)$ are the Milnor numbers.
We tried to construct a deformation f_t of f_0 where the multiplicity $\mathrm{mult}\,(f_t)$ drops but the Milnor number $\mu(f_t)$ is constant.
Our candidates ($a, b, c \in \mathbb{N}$) came from a heuristical investigation of the Newton diagram, one being the following series:

$$f_t = x^a + y^b + z^{3c} + x^{c+2}y^{c-1} + x^{c-1}y^{c-1}z^3 + x^{c-2}y^c(y^2 + tx)^2, \ a, b, c \in \mathbb{N}.$$

Obviously, the multiplicity drops. Computing μ with SINGULAR, we obtain for $(a, b, c) = (37, 27, 6)$: $\mu(f_0) = 4840$, $\mu(f_t) = 4834$, thus f_0 and f_t are (unfortunately) not topologically equivalent.

Since the Milnor numbers of possible counter examples have to be very big, we need an extremely efficient implementation of standard bases. For this, the "highest corner" method of Greuel and Pfister (1996) was essential.

Trying many other classes of examples, we did not succeed in finding a counter example. However, an analysis of the examples led to the following

Partial proof of Zariski's conjecture (Greuel and Pfister, 1996):
Zariski's conjecture is true for deformations of the form

$$f_t = g_t(x, y) + z^2 h_t(x, y), \quad \operatorname{mult}(g_t) < \operatorname{mult}(f_0).$$

There is also an invariant characterisation of the deformations of the above kind. The general conjecture is, up to today, still open.

7.3. CURVES WITH MAXIMAL NUMBER OF SINGULARITIES

Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be an irreducible projective curve of degree d and $f(x, y) = 0$ a local equation for the germ (C, z). Let $\mu(C, z) = \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(f_x, f_y)$ be the Milnor number of C at z.

Since the genus of C, $g(C) = \frac{(d-1)(d-2)}{2} - \delta(C)$ is non–negative (where $\delta(C) = \sum_{z \in C} \delta(C, z)$, $\delta(C, z) = \dim_{\mathbb{C}} \bar{R}/R$, $R = \mathbb{C}\{x, y\}/\langle f \rangle$ and $\bar{R}$ the normalisation of R), C can have, at most, $(d-1)(d-2)/2$ singularities.

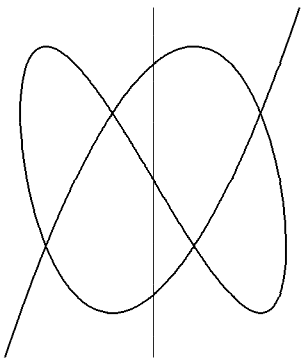
It is a classical and interesting problem, which is still in the centre of theoretical research, to study the variety $V = V_d(S_1, \ldots, S_r)$ of (irreducible) curves $C \subset \mathbb{P}^2_{\mathbb{C}}$ of degree d having exactly r singularities of prescribed (topological or analytical) type $S_1, \ldots, S_r$. Among the most important questions are:

Is $V \neq \emptyset$ (existence problem)?
Is V irreducible (irreducibility problem)?
Is V smooth of expected dimension (T–smoothness problem)?

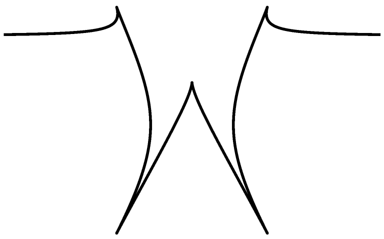
A complete answer is only known for nodal curves, that is, for $V_d(r) = V_d(S_1, \ldots, S_r)$ with S_i ordinary nodes (A_1–singularities):

Severi (1921): $V_d(r) \neq \emptyset$ and T–smooth $\Leftrightarrow r \leq \frac{(d-1)(d-2)}{2}$.
Harris (1985): $V_d(r)$ is irreducible (if $\neq \emptyset$).

Even for cuspidal curves a sufficient and necessary answer to any of the above questions is unknown.



A 4-nodal plane curve of degree 5, with equation $x^5 - \frac{5}{4}x^3 + \frac{5}{16}x - \frac{1}{4}y^3 + \frac{3}{16}y = 0$, which is a deformation of $E_8 \ : \ x^5 - y^3 = 0$.



A plane curve of degree 5 with 5 cusps, the maximal possible number. It has the equation $\frac{129}{8}x^4y - \frac{85}{8}x^2y^3 + \frac{57}{32}y^5 - 20x^4 - \frac{21}{4}x^2y^2 + \frac{33}{8}y^4 - 12x^2y + \frac{73}{8}y^3 + 32x^2 = 0$.

Concerning arbitrary (topological types of) singularities, we have the following existence theorem, which is, with respect to the exponent of d, asymptotically optimal.

Theorem: (Greuel, Lossen and Shustin, 1998; Lossen, 1999).
$V_d(S_1,\ldots,S_r)\neq\emptyset$ if $\sum_{i=1}^r \mu(S_i)\leq \dfrac{(d+2)^2}{46}$ and two additional conditions for the five "worst" singularities.

In case of only one singularity we have the slightly better sufficient condition for existence,

$$\mu(S_1)\leq\frac{(d-5)^2}{29}.$$

The theorem is just an existence statement, the proof gives no hint how to produce any equation. Having a method for constructing curves of low degree with many singularities, Lossen was able to produce explicit equations. In order to check his construction and improve the results, he made extensive use of SINGULAR to compute standard bases for global as well as for local orderings. One of his examples is the following:

Example: (Lossen, 1999) The irreducible curve with affine equation $f(x,y)=0$,

$$\begin{aligned} f(x,y) = y^2 - 2y(x^{10} &+ \frac{1}{2}x^9y^2 - \frac{1}{8}x^8y^4 + \frac{1}{16}x^7y^6 - \frac{5}{128}x^6y^8 + \frac{7}{256}x^5y^{10} \\ &- \frac{21}{1024}x^4y^{12} + \frac{33}{2048}x^3y^{14} - \frac{429}{32768}x^2y^{16} + \frac{715}{65536}xy^{18} \\ &- \frac{2431}{262144}y^{20}) + x^{20} + x^{19}y^2 \end{aligned}$$

has degree 21 and an A_{228}–singularity $(x^2-y^{229}=0)$ as its only singularity.

In order to verify this, one may proceed, using SINGULAR, as follows:

```
ring s = 0,(x,y),ds;
poly f = y2-2x10y-x9y3+1/4x8y5-1/8x7y7+5/64x6y9-7/128x5y11+21/512x4y13
        -33/1024x3y15+429/16384x2y17+x20-715/32768xy19+x19y2+2431/131072y21;
matrix Hess = jacob(jacob(f));          //the Hessian matrix of f
print(subst(subst(Hess,x,0),y,0));      //the Hessian matrix for x=y=0
==> 0,0,
==> 0,2
vdim(std(jacob(f)));                   //the Milnor number of f
==> 228
```

Since the rank of the Hessian at 0 is 1, f has an A_k singularity at 0; it is an A_{228} singularity since the Milnor number is 228. To show that the projective curve C defined by f has no other singularities, we have to show that C has no further singularities in the affine part and no singularity at infinity. The second assertion is easy, the first follows from

$$dim_{\mathbb{C}}(K[x,y]_{\langle x,y\rangle}/\langle \mathrm{jacob}(f),f\rangle = \dim_{\mathbb{C}}(K[x,y]/\langle \mathrm{jacob}(f),f\rangle,$$

confirmed by SINGULAR:

```
vdim(std(jacob(f)+f));
==> 228                    //multiplicity of Sing(C) at 0 (local ordering)
ring r = 0,(x,y),dp;
poly f = fetch(s,f);
vdim(std(jacob(f)+f));
==> 228                    //multiplicity of Sing(C)      (global ordering)
```

8. What else is needed

In this survey I could only touch on a few topics where computer algebra has contributed to mathematical research. Many others have not been mentioned, although there exist powerful algorithms and efficient implementations. In the first place, the computation of invariant rings for group actions of finite (Sturmfels, 1993; Kemper, 1996; Decker and De Jong, 1998), reductive (Derksen, 1997) or some uni–potent (Greuel, Pfister and Schönemann, 1990–1998) groups belong here. Computation of invariants have important applications for explicit construction of moduli spaces, for example, for vector bundles or for singularities (Frühbis-Krüger, 2000; T. Bayer, 2000) but also for dynamical systems with symmetries (Gatermann, 1999). Libraries for computing invariants are available in SINGULAR. Available is also the Puiseux expansion (even better, the Hamburger–Noether expansion, cf. Lamm (1999) for description of an implementation) of plane curve singularities. The latter is one of the few examples of an algorithm in algebraic geometry where Gröbner bases are not needed.

The applications of computer algebra and, in particular, of Gröbner bases in projective algebraic geometry are so numerous that I can only refer to the textbooks of Cox–Little–O'Shea, Eisenbud and Vasconcelos and the literature cited there. The applications include classification of varieties and vector bundles, cohomology, moduli spaces and fascinating problems in enumerative geometry.

However, there are also some important problems for which an algorithm is either not known or not yet implemented (for further open problems see also Eisenbud (1993)):

1 Resolution of singularities
This is one of the most important tools for treating singular varieties. At least three approaches seem to be possible:
For surfaces we have Zariski's method of successive normalisation and blowing up points and the Hirzebruch–Jung method of resolving the discriminant curve of a projection.
For arbitrary varieties, new methods of Bierstone, Milman and Villamajor provide a constructive approach to resolution in the spirit of Hironaka. First attempts in this direction have been made by Schicho.

2 Computation in power series rings
This is a little vague since I do not mean to actually compute with infinite power series, the input should be polynomials. However, it would be highly desirable to make effective use of the Weierstraß preparation theorem. This is related to the problem of elimination in power series rings.
Moreover, no algorithm seems to be known to compute an algebraic representative of the semi-universal deformation of an isolated singularity (which is known to exist).
Also, I do not know any algorithm for Hensel's lemma.

3 Dependence of parameters
In this category falls, at least principally, the study of Gröbner bases over rings. This has, of course, been studied, cf. Adams and Loustaunou (1994), Kalkbrenner (1998), but I still consider the dependence of Gröbner bases on parameters as an unsolved problem (in the sense of an intrinsic or predictable description, if it exists).
In many cases, one is interested in finding equations for parameters describing precisely the locus where certain invariants jump. This is related to the above problem since Gröbner bases usually only give a sufficient but not necessary answer.
Mainly in practical applications of Gröbner bases to "symbolic solving", parameters are real or complex numbers. It would then be important to know, for which range of the parameters the symbolic solution holds.

4 Symbolic–numeric algorithms
The big success of numerical computations in real life problems seems to show that symbolic computation is of little use for such problems. However, as is well–known, symbolic preprocessing of a system of polynomial (even ordinary and partial differential) equations may not only lead to much better conditions for the system to be solved numerically but even make numerical solving possible.
There is continuous progress in this direction, cf. Cox, Little and O'Shea (1998), Möller (1998), Verschelde (1999), not only by Gröbner basis methods. A completely different approach via multivariate resultants (cf. Canny and Emiris (1997)) has become favourable to several people due to the new sparse resultants by Gelfand, Kapranov and Zelevinksi (1994). However, an implementation in SINGULAR (cf. Wenk (1999), Hillebrand (1999))

does not show superiority of resultant methods, at least for many variables against triangular set methods of either Lazard or Möller. Nevertheless, much more has to be done.
The main disadvantage of symbolic methods in practical, real life applications is its complexity. Even if a system is able to return a symbolic answer in a short time, this answer is often not humanly interpretable. Therefore, a symbolic simplification is necessary, either before, during, or after generation. Of course, the result must still be approximately correct.
This leads to the problem of validity of "simplified" symbolic computation. A completely open subproblem is the validity resp. error estimation of Gröbner basis computations with floating point coefficients.
The simplification problem means providing simple and humanly understandable symbolic solutions which are approximately correct for numerical values in a region which can be specified. This problem belongs, in my opinion, perhaps to the most important ones in connection with applications of computer algebra to industrial and economical problems.

5 Non–commutative algorithms
Before Gröbner bases were introduced by Buchberger, the so–called Ritt–Wu method, cf. Ritt (1950), was developed for symbolic computation in non–commutative rings of differential operators. However, nowadays, commutative Gröbner bases are implemented in almost every major computer algebra system, whilst only few systems provide non–commutative algorithms. Standard bases for some non–commutative structures have been implemented in the system FELIX (Apel and Klaus, 1991) as well as in an experimental version in SINGULAR; the system Bergman and an extension called Anick can compute Gröbner bases and higher syzygies in the non–commutative case.
Highly desirable are effective implementations for non–commutative Gröbner bases in the Weyl algebra, the Grassmannian, for D–modules or the enveloping algebra of a finite dimensional Lie algebra (the general theory being basically understood, cf. Mora (1989), Ufnarovski (1998), Apel (1998)).
The recent textbook of Saito, Sturmfels and Takayama (1999) shows a wide variety of algorithms for modules over the Weyl algebra and D–modules for which an efficient implementation is missing.
But even classical algebraic geometry, as was shown, for example, by Kontsevich and Manin (1994), has a natural embedding into non–commutative algebraic geometry. A special case is known as quantisation, a kind of non–commutative deformation of a commutative algebra.
Providing algorithms and implementations for the use of computer algebra in non–commutative algebraic geometry could become a task and challenge for a new generation of computer algebra systems.

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