THREE ALGORITHMS IN ALGEBRAIC GEOMETRY, CODING THEORY AND SINGULARITY THEORY

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Introduction

Algorithmic and computational aspects have become a major, still growing issue in mathematical research and teaching. This has various deeper reasons in the cultural and technological development of today's society but also quite practical reasons. One of these is certainly the existence and maturity of software systems which, via implemented algorithms, provide easy access to hard and sometimes sophisticated computations, assisting and supporting mathematical research.

In this article, we describe three algorithms in algebraic geometry, coding theory and singularity theory, which are new, resp. have new ingredients. The first one describes how to compute the normalization of an affine reduced ring, an ideal defining the non-normal locus and, as an application, the integral closure of an ideal. The second is devoted to the computation of the places of a projective plane curve defined over a finite field, and the computation of bases of adjoint forms and of the linear system of a given rational divisor on the normalization of the curve. Finally, the third one provides a method to compute the V-filtration, the monodromy and the singularity spectrum of an arbitrary isolated hypersurface singularity.

All three algorithms require non-trivial up to deep mathematical knowledge and go beyond foundational algorithms in computer algebra. Indeed, one of the purposes of this note is to show that highly complex mathematical objects can nowadays be represented in a computer and, thus, can be used in mathematical research on a higher level than ever before. All the algorithms described in this paper are implemented in the computer algebra system Singular [17] as a free service to the mathematical community.

The normalization algorithm is based on an old criterion of Grauert and Remmert and has already been published [6, 5]. From this it is not difficult to derive the principle for algorithms to compute the non-normal locus and the integral closure of an ideal, however, the concrete description and its realization appear to be new. The proposed algorithm for computing the places of plane curves is based on the Hamburger-Noether development and has been described on a theoretical level in [3], as well as the Brill-Noether algorithm for computing bases of linear systems. Since then, this algorithm has been implemented, together with a full coding and decoding algorithm, and we mainly concentrate on new algorithmic and computational aspects. It should be mentioned that the construction of AG codes, using quadratic transformations instead of Hamburger-Noether expansions, has been described and implemented before [28, 19]. Finally, the algorithm to compute the V-filtration and the singularity spectrum is very recent, a theoretical description of parts of it are to be published in [37]. Here, we give a short description of the theory together with a description of some computational aspects. A paper with full details will be published by the third author.

1. Integral closure of rings and ideals

In this section we present an algorithm to compute, for a reduced affine ring A, the normalization, that is, the integral closure of A in its total ring of fractions. This algorithm can be used to describe an algorithm for computing the integral closure of an ideal I in A.

The normalization of a ring is an important construction in commutative algebra and algebraic geometry, as well as the integral closure of an ideal. Both have many applications in commutative algebra, algebraic geometry and singularity theory, e.g., in the theory of equisingularity and for resolution of singularities. Hence it is desirable to have a sufficiently good implementation of the algorithms which can compute interesting examples. Such an implementation is distributed with Singular, version 2.0, [17].

The algorithm for the normalization is based on a criterion of Grauert and Remmert and was first described in [6], see also [5]. The Grauert-Remmert criterion provides an algorithm to compute the non-normal locus of an affine ring, which we also describe. Both seem to be the only known general algorithms. To compute the integral closure of an ideal I, we use the normalization of the Rees algebra of I together with several extra procedures, which make this basically a new algorithm.

A good reference for computational aspects in connection with integral closure is found in the textbooks [44, 45] and in the articles [43, 4]. Other references are [39] and [10] or can be found in [44, 45].

Let $Q(A)$ denote the total ring of fractions of A and $\overline{A}$ the integral closure of A in $Q(A)$. There have been many attempts to construct a bigger ring A', $A \subset A' \subset Q(A)$, which is finite over A and then continuing with A', in order to approximate $\overline{A}$. The problem of this approach is to know when to stop, that is, to have an effective criterion for a ring to be normal. It had escaped the computer algebra community that such a criterion has been known for more than thirty years, having been discovered by Grauert and Remmert [12]. It was rediscovered by De Jong [6] and says that A is normal if and only if the natural embedding $A \subset \mathrm{Hom}_A(J, J)$, where J denotes the ideal of the singular locus, is an isomorphism. To be able to continue with $A' = \mathrm{Hom}_A(J, J)$ we must present A' as a polynomial ring modulo some ideal, together with the embedding $A \hookrightarrow A'$, which is not difficult. By the theorem of Grauert and Remmert, we know when to stop to reach the normalization of A (for affine rings the algorithm stops by a classical theorem of E. Noether).

If $I \subset A$ is an ideal, then it is well-known that the Rees algebra

$$\mathcal{R}(I) = \bigoplus_{k \geq 0} I^k t^k \subset A[t]$$

of I has, as normalization, $\overline{\mathcal{R}(I)} = \bigoplus_{k\geq 0} \overline{I^k} t^k$. That is, the component of t-degree 1 of $\overline{\mathcal{R}(I)}$ is the integral closure $\overline{I}$ of I in A. However, to obtain the component of t-degree k of $\overline{\mathcal{R}(I)}$ we need a careful analysis of the morphism $\mathcal{R}(I) \subset \overline{\mathcal{R}(I)} \subset A[t]$. The embedding $\overline{\mathcal{R}(I)} \subset A[t]$ is extra information which has to be added to the normalization algorithm.

Unfortunately, this algorithm computes too much, namely all $\overline{I^k}$ and, therefore, its application is restricted only to examples of moderate size. However, already a first implementation in SINGULAR, cf. [22, 23], shows that we can compute interesting examples (and it is the only existing implementation). It is still an open problem to find a better algorithm, not using the Rees algebra.

1.1. **Ring normalization.** Let $A \subset B$ be a ring extension and $I \subset A$ an ideal. Recall that $b \in B$ is called (strongly) *integral over* I if b satisfies a relation

$$b^n + a_1 b^{n-1} + \ldots + a_n = 0 \text{ with } a_i \in I^i .$$

The set $C(I,B) = \{b \in B \mid b \text{ is integral over } I\}$ is called the *integral closure of* I in B, it is a $C(A,B)$-module, where $C(A,B)$, the integral closure of A in B, is a ring.

We are interested in the two most interesting cases:

- A reduced, $B = Q(A)$ the total quotient ring of A, and $I = A$. In this case $I^i = I$ and $C\big(A, Q(A)\big) =: \overline{A}$ is the *normalization* of A.
- $A = B$ and $I \subset A$ arbitrary. Then $C(I,A) =: \overline{I}$ is the integral closure of I in A.

Let us consider the normalization first.

Lemma 1.1 (Key-lemma)**.** *Let A be a reduced Noetherian ring and $J \subset A$ an ideal containing a non-zerodivisor u of A. Then there are natural inclusions of rings*

$$A \subset \operatorname{Hom}_A(J,J) \cong \frac{1}{u}(uJ : J) \subset \overline{A}\,,$$

and inclusions $\operatorname{Hom}_A(J,J) \subset \operatorname{Hom}_A(J,A) \cap \overline{A} \subset \operatorname{Hom}_A\big(J, \sqrt{J}\big)$ *of A-modules. Here $uJ : J = \big\{h \in A \,|\, hJ \subset uJ\big\}$.*

Proof. If $\varphi \in \operatorname{Hom}_A(J,A)$, then $\frac{\varphi(u)}{u}$ is independent of the non-zerodivisor u and, hence, $\varphi \mapsto \frac{\varphi(u)}{u}$ defines an embedding

$$\operatorname{Hom}_A(J,A) \xrightarrow{\cong} \{h \in Q(A) \mid hJ \subset A\} \hookrightarrow Q(A)\,.$$

The inclusion $A \subset \operatorname{Hom}_A(J,J) \cong \frac{1}{u}(uJ : J) = \frac{1}{u}\{h \in Q(A) \mid hJ \subset uJ\}$ is given by the multiplication with elements of A. To see that the image is contained in $\overline{A}$, consider $\varphi \in \operatorname{Hom}_A(J,J)$. The characteristic polynomial of φ defines (by Cayley-Hamilton) an integral relation for φ. To see the last inclusion, consider $h \in \overline{A}$ such that $hJ \subset A$, and let $h^n + a_1 h^{n-1} + \cdots + a_n = 0$, $a_i \in A$, be an integral relation for h. For a given $g \in J$ multiply the relation for h with g^n. This shows that $(gh)^n \in J$, hence $gh \in \sqrt{J}$. □

Since A is normal if and only if the localization A_P is normal for all $P \in \operatorname{Spec} A$, we define the *non-normal locus* of A as

$$\operatorname{NN}(A) := \{P \in \operatorname{Spec} A \mid A_P \text{ is not normal}\}\,.$$

It is easy to see that $\operatorname{NN}(A) = V(C)$ where $C = \operatorname{Ann}_A(\overline{A}/A)$ is the *conductor* of A in $\overline{A}$, in particular, $\operatorname{NN}(A)$ is closed in $\operatorname{Spec} A$. However, since we cannot yet compute $\overline{A}$, we cannot compute C either.

The following proposition is the basis for the algorithm to compute the normalization as well as for an algorithm to compute an ideal with zero set $\operatorname{NN}(A)$. It is basically due to Grauert and Remmert, [12].

Proposition 1.2 (Criterion for normality)**.** *Let A be a reduced Noetherian ring and $J \subset A$ an ideal satisfying*

1. *J contains a non-zerodivisor of A,*
2. *$J = \sqrt{J}$,*
3. *$\mathrm{NN}(A) \subset V(J)$.*

Then $A = \overline{A}$ if and only if $A = \mathrm{Hom}_A(J,J)$.

An ideal $J \subset A$ satisfying 1.–3. is called a *test ideal* for the normalization.

Proof. If $A = \overline{A}$ then $\mathrm{Hom}_A(J,J) = A$, by Lemma 1.1. For the converse, notice that 3. implies $J \subset \sqrt{C}$, hence there exists a minimal $d \geq 0$ such that $J^d \in C$, that is, $J^d\overline{A} \subset A$. If $d > 0$, choose $h \in \overline{A}$ and $a \in J^{d-1}$ such that $ha \notin A$. Since $ah \in \overline{A}$ and $ahJ \subset hJ^d \subset A$ we have $ah \in \mathrm{Hom}_A(J,A) \cap \overline{A}$ and hence, using 2., $ah \in \mathrm{Hom}_A(J,J)$, by Lemma 1.1. By assumption $\mathrm{Hom}_A(J,J) = A$ and, hence, $ah \in A$. This is a contradiction, and we conclude $d = 0$ and $A = \overline{A}$. □

Remark 1.3. As the proof shows, condition 2. can be weakened to

2'. $\mathrm{Hom}_A(J,J) = \mathrm{Hom}_A(J,A) \cap \overline{A}$.

However, this cannot be used in practice, since we do not know $\overline{A}$, while, on the other hand, we can always pass from J to $\sqrt{J}$ without violating the conditions 1 and 3, and $\sqrt{J}$ is computable.

Corollary 1.4. *Let $J \subset A$ be as in Prop. 1.2 and $I_{NN} := Ann_A\big(\mathrm{Hom}_A(J,J)\big)$. Then $\mathrm{NN}(A) = V(I_{NN})$.*

Proof. This follows from 1.2, 1.1 and the fact that the operations which define the annihilator are compatible with localization. □

Having a test ideal J and a non-zerodivisor $u \in J$ of A, we can compute A-module generators of $\mathrm{Hom}_A(J,J) \cong uJ : J$ and A-module generators for $I_{NN} = \mathrm{Ann}_A\big(\mathrm{Hom}_A(J,J)\big) \cong \langle u \rangle : (uJ : J)$, since we can compute ideal quotients using Gröbner basis methods, cf. [15].

Let us describe the *ring structure* of $\mathrm{Hom}_A(J,J)$. For this, let $u_0 = u$, $u_1, \dots, u_s$ be generators of $uJ : J$ as A-module, and let $(\alpha_0^i, \dots, \alpha_s^i)$ be generators of the module of syzygies of $u_0, \dots, u_s$. Since $\mathrm{Hom}_A(J,J)$ is a ring, we have $u_i \cdot u_j = \sum_{\ell=0}^s \beta_\ell^{ij} u_\ell$, $1 \leq i,j \leq s$ for certain $\beta_\ell^{ij} \in A$, the quadratic relations between the u_i. Define $\mathrm{Ker} \subset A[t_1, \dots, t_s]$ as the ideal generated by the linear and quadratic relations,

$$\alpha_0^i + \alpha_1^i t_1 + \cdots + \alpha_s^i t_s\,, \quad t_i t_j - (\beta_0^{ij} + \beta_1^{ij} t_1 + \cdots + \beta_s^{ij} t_s)\,.$$

We get an isomorphism $\mathrm{Hom}_A(J,J) \cong uJ : J \cong A[t_1, \dots, t_s]/\,\mathrm{Ker}$ of A-algebras, by sending t_i to u_i. This presentation is needed to continue the normalization algorithm.

To compute I_{NN}, we only need the A-module structure of $uJ : J$.

1.2. **Test ideals.** It remains to find a test ideal. For this we consider the singular locus

$$\mathrm{Sing}(A) = \{P \in \mathrm{Spec}\, A \mid A_P \text{ is not regular}\}\,.$$

Since every regular local ring is normal, $\mathrm{NN}(A) \subset \mathrm{Sing}(A)$. For general Noetherian rings, however, $\mathrm{Sing}(A)$ may not be closed in the Zariski topology. Therefore, we pass to more special rings.

Let $S = K[x_1, \dots, x_n]$ and $A = S/I$ be an affine ring where K is a perfect field. If A is equidimensional of codimension c, that is, all minimal primes P of I have the same height c, then the *Jacobian ideal* of I defines $\operatorname{Sing}(A)$. That is, if $I = \langle f_1, \dots, f_k \rangle$ and

$$J = \left\langle c\text{-minors of } \left(\frac{\partial f_i}{\partial x_j}\right),\ f_1, \dots, f_k \right\rangle \subset S$$

is the Jacobian ideal of I, then $\operatorname{Sing}(A) = V(J)$. If, on the other hand, A is not equidimensional, then $V(J)$ may be strictly contained in $\operatorname{Sing}(A)$, if we define J as above with c the minimal height of minimal primes of I. In this case we need another ideal. There are several alternatives to compute an ideal I_{Sing} with $V(I_{\text{Sing}}) = \operatorname{Sing}(A)$. Either we compute an equidimensional decomposition $I = \bigcap_i I_i$, [8, 15], of I, compute the Jacobian ideal J_i for each equidimensional ideal I_i and compute the ideal describing the intersection of any two equidimensional parts. The same works for a primary decomposition, [11, 8, 15]) instead of an equidimensional decomposition.

We can avoid an equidimensional, resp. primary, decomposition if we compute the ideal of the non-free locus of the module of *Kähler differentials*,

$$\Omega^1_{A/K} = \Omega^1_{S/K} \Big/ \left(\sum_{i=1}^{k} f_i \Omega^1_{S/K} + \sum_{i=1}^{k} S df_i \right),$$

where $\Omega^1_{S/K} = \bigoplus_{i=1}^n S dx_i$. $\Omega^1_{A/K}$ is isomorphic to A^n modulo the submodule generated by the rows of the Jacobian matrix of $(f_1, \dots, f_k)$, hence it is finitely presented by the transpose of the Jacobian matrix.

For any finitely presented A-module M we can compute the non-free locus

$$\operatorname{NF}(M) = \{P \in \operatorname{Spec} A \mid M_P \text{ is not } A_P\text{-free}\}$$

just by Gröbner basis and syzygy computations, cf. [15].

Let I_{Sing} be an ideal defining the singular locus of $A = S/I$. Since A is reduced, I_{Sing} contains a non-zerodivisor of A. Indeed, a general linear combination u of the generators of I_{Sing} will be a non-zerodivisor. Hence, any radical ideal of S which contains I and u will be a test ideal for normality. Two extreme choices for test ideals are $\sqrt{I_{\text{Sing}}}$ or $\sqrt{\langle I, u \rangle}$.

Since the radical of an ideal in an affine ring can be computed, [8, 25, 15], we have all ingredients to compute the normalization of A.

In the remaining part of this section, we describe algorithms to compute

- the *normalization* $\overline{A}$ of A, that is, we represent $\overline{A}$ as affine ring and describe the map $A \hookrightarrow \overline{A}$,
- generators for an ideal $I_{NN} \subset A$ describing the *non-normal locus*, that is, $V(I_{NN}) = \operatorname{NN}(A) = \{P \in \operatorname{Spec} A \mid A_P \text{ is not normal}\}$,
- for any ideal $I \subset A$, generators for the *integral closure* $\overline{I}$ of I in A.

1.3. **Normalization algorithm.** Let us first describe the algorithm to compute the non-normal locus. K denotes a perfect field.

1.3.1. *Computing the non-normal locus.*

Input: $f_1, \dots, f_k \in S = K[x_1, \dots, x_n]$, $I := \langle f_1, \dots, f_k \rangle$.
We assume that I is a radical ideal.

Output: Generators for I_{NN} such that $V(I_{NN}) = \operatorname{NN}(S/I)$.

1. Compute an ideal I_{Sing} as described in Sect. 1.2.
2. Compute a non-zerodivisor $u \in I_{\text{Sing}}$: choose an S-linear combination u of the generators of I_{Sing} and test if $(I : u) := \{g \in S \mid gu \in I\}$ is zero, by using that u is a non-zerodivisor iff $(I : u) = 0$.
   A sufficiently general linear combination gives a non-zerodivisor.
3. Compute a test ideal J, e.g., $J = \sqrt{\langle u, I\rangle}$ or $J = \sqrt{I_{\text{Sing}}}$.
4. Compute generators $g_1, \dots, g_\ell$ for $\langle u, I\rangle : \big((uJ+I) : J\big)$ as S-module.
5. Return $\{g_1, \dots, g_\ell\}$.

1.3.2. *Computing the normalization.* The idea for computing the normalization of $A = S/I$ is as follows:

- We construct an increasing sequence of rings
$$A \subset A_1 \subset A_2 \subset \cdots \subset A_k \subset A_{k+1} \subset \cdots \subset Q(A)$$
with $J_0 \subset A_0 := A$ a test ideal for A, $A_i := \operatorname{Hom}_{A_{i-1}}(J_{i-1}, J_{i-1})$, and $J_i \subset A_i$ a test ideal for A_i, $i \geq 1$.
- If $A_k = A_{k+1}$ then $\overline{A} = A_k$.

For performance reasons we do not look for a non-zerodivisor in J_i but choose any non-zero element u. If u is a zerodivisor then it gives a splitting of the ring which makes the subsequent computations easier.

We obtain the following (highly recursive) algorithm for computing the normalization:

Input: $f_1, \dots, f_k \in S = K[x_1, \dots, x_n]$, $I = \langle f_1, \dots, f_k\rangle$
We assume that I is a radical ideal.

Output: Polynomial rings $S_1, \dots, S_\ell$, ideals $I_j \subset S_j$, and maps $\pi_j : S \to S_j$ such that $S/I \to S_1/I_1 \oplus \cdots \oplus S_\ell/I_\ell$, induced by $(\pi_1, \dots, \pi_\ell)$ is the normalization map of S/I.

1. Compute an ideal I_{Sing} describing the singular locus.
2. Compute the radical $J = \sqrt{I_{\text{Sing}}}$.
3. Choose $u \in J \setminus \{0\}$ and compute $I : u$. If $I : u = I$ go to 4 (then u is a non-zerodivisor of S/I). Otherwise, set $R = A/\langle u\rangle \oplus A/(I : u)$ and go to 6.
4. Compute $R := \operatorname{Hom}_A(J, J)$ and $\pi : A \to R$.
5. If $A = R$ then return (R, π), otherwise go to 1.
6. Suppose $R = R_1/I_1 \oplus \cdots \oplus R_\ell/I_\ell$. Then, for each $i = 1, \dots, \ell$, set $A = R_i/I_i$ and go to 1.

1.4. **Integral closure algorithm.** Let A be a ring, $I \subset A$ an ideal. We propose an algorithm to compute $\overline{I} = \{b \in A \mid b \text{ is integral over } I\}$, the *integral closure* of I. I is called *integrally closed* if and only if $I = \overline{I}$. It is called *normal* if $I^k = \overline{I^k}$ for all $k > 0$. Note that $I \subset \overline{I} \subset \sqrt{I}$. We are mainly interested in the case $A = K[x_1, \dots, x_n]$.

In the following, we describe an algorithm to compute $\overline{I^k}$ for all k, simultaneously. Consider the *Rees algebra* $\mathcal{R}(I) = \bigoplus_{k\geq 0} I^k t^k \subset A[t]$, and let $\widetilde{\mathcal{R}(I)}$ denote the integral closure of $\mathcal{R}(I)$ in $A[t]$. Then
$$\widetilde{\mathcal{R}(I)} = \bigoplus_{k\geq 0} \overline{I^k} t^k \subset A[t]\,.$$

If A is normal, then $A[t]$ is normal and hence, the normalization of $\mathcal{R}(I)$, that is, the integral closure of $\mathcal{R}(I)$ in $Q(\mathcal{R}(I))$, satisfies

$$\overline{\mathcal{R}(I)} = \widetilde{\mathcal{R}(I)} = \bigoplus_{k\geq 0} \overline{I^k} t^k .$$

Hence, computing the normalization of $\mathcal{R}(I)$ provides the integral closure of $\overline{I^k}$ for all k.

To be specific, let $A = K[\boldsymbol{x}] = K[x_1, \dots, x_n]$, $I = \langle f_1, \dots, f_k\rangle \subset A$ with K a perfect field. Then

$$\mathcal{R}(I) \;=\; K[\boldsymbol{x}, tf_1, \dots, tf_k] \;\underset{\varphi}{\overset{\cong}{\longleftarrow}}\; K[\boldsymbol{x}, U_1, \dots, U_k]/(\mathrm{Ker}\,\varphi)$$

where $\varphi : K[\boldsymbol{x}, \boldsymbol{U}] \to K[\boldsymbol{x}, t]$ maps $x_i \mapsto x_i$, $U_j \mapsto tf_j$. $\mathrm{Ker}\,\varphi$ can be computed by eliminating t from

$$J := \langle U_1 - tf_1, \dots, U_k - tf_k\rangle \subset K[\boldsymbol{x}, \boldsymbol{U}, t] ,$$

that is, $\mathrm{Ker}\,\varphi = J \cap K[\boldsymbol{x}, \boldsymbol{U}]$. For the integral closure of I we need to compute

$$\begin{tikzcd}
\mathcal{R}(I) \;\cong & K[\boldsymbol{x}, \boldsymbol{U}]/(\mathrm{Ker}\,\varphi) \arrow[r, "\varphi"] \arrow[d, hook] & K[\boldsymbol{x}, t] \\
Q\big(\mathcal{R}(I)\big) \;\supset\; \overline{\mathcal{R}(I)} \;= & K[T_1, \dots, T_s]/I' \arrow[ur, "{T_i \mapsto \frac{\varphi(a_i)}{\varphi(b_i)},\ T_i = \frac{a_i}{b_i} \in Q(\mathcal{R}(I))}"'] &
\end{tikzcd}$$

This means that we compute $\overline{\mathcal{R}(I)}$ as an affine ring $K[\boldsymbol{T}]/I'$ and, in each inductive step during the computation of $\overline{\mathcal{R}(I)}$, we also compute the map from the intermediate ring to $K[\boldsymbol{x}, t]$.

The algorithm then reads as follows:

Input: $f_1, \dots, f_\ell \in K[x_1, \dots, x_n]$, $k \geq 1$ an integer, $I := \langle f_1, \dots, f_\ell\rangle$.
Output: Generators for $\overline{I^k} \subset K[x_1, \dots, x_n]$.

1. Compute the Rees algebra $\mathcal{R}(I) \subset K[\boldsymbol{x}, t]$.
2. Compute the normalization $\overline{\mathcal{R}(I)}$, together with maps φ, ψ, so that

$$\begin{tikzcd}
 & \mathcal{R}(I) \arrow[r, hook, "\varphi"] \arrow[d, hook] & K[\boldsymbol{x}, t] \\
Q\big(\mathcal{R}(I)\big) \;\supset & \overline{\mathcal{R}(I)} \arrow[r, equal] & K[T_1, \dots, T_s]/J \arrow[u, hook, "\psi"']
\end{tikzcd}$$

   commutes.
3. Determine $a_i, b_i \in \mathcal{R}(I)$, so that $T_i = \frac{a_i}{b_i} \in Q\big(\mathcal{R}(I)\big)$ compute $\frac{\varphi(a_i)}{\varphi(b_i)} \in K[\boldsymbol{x}, t]$ (indeed, we find a universal denominator $b = b_i$ for all i).
4. Determine generators $g_1, \dots, g_s$ of the $K[\boldsymbol{x}]$-ideal which is mapped to the component of t-degree k of the subalgebra $\psi\big(\overline{\mathcal{R}(I)}\big) \subset K[\boldsymbol{x}, t]$.
5. Return $g_1, \dots, g_s$.

The algorithms described above are implemented in SINGULAR and contained in the libraries `normal.lib` [16] and `reesclos.lib` [23] contained in the distribution of SINGULAR 2.0 [17]. Similar procedures can be used to compute the conductor ideal of A in $\overline{A}$. An implementation will be available soon.

1.5. Examples.

1.5.1. *Ring normalization.* We first load the library `normal.lib`

```
LIB "normal.lib";
```

We define the ring and an ideal, describing two transversal cusps.

```
ring S  = 0,(x,y),dp;
ideal I = (x2-y3)*(x3-y2);
```

If the printlevel is sufficiently high, the algorithm will display intermediate results and pause during the computation until the user hits the return button in order to be able to follow what's going on. Here we will reproduce only part of the comments and slightly change the Singular output.

```
printlevel = 5;
```

Now we start the computation using the procedure `normal` from `normal.lib`:

```
list nor = normal(I);
```

The first normalization loop starts with $\langle x^3y^3 - x^5 - y^5 + x^2y^2\rangle$, and computes the radical of the singular locus as

$$J = \langle xy^2 - y,\, x^2y - x,\, y^4 - x,\, x^4 - y\rangle\,,$$

chooses the non-zerodivisor $u = 3x^2y^3 - 5x^4 + 2xy^2$ in J, computes

$$\begin{aligned} uJ : J \quad = \quad & \langle 3x^2y^3 - 5x^4 + 2xy^2,\, 2x^5 - 3y^5 + x^2y^2,\, 3y^6 - 5x^4 + 2xy^2, \\ & \qquad xy^5 - x^3y^2,\, x^4y^2 - xy^4\rangle\,, \end{aligned}$$

and checks that $A \neq \mathrm{Hom}_A(J,J)$. Therefore the ring structure of $\mathrm{Hom}_A(J,J)$ has to be computed. The result is the affine ring A_1 with variables $T_1,\dots,T_4$ (after eliminating linear equations) modulo the ideal

```
==> T(3)*T(4)-T(4)
==> T(1)^4+4*T(1)^2*T(2)*T(4)+4*T(2)^2*T(4)^2
          -T(2)*T(3)+T(2)
==> T(1)^2*T(2)+2*T(2)^2*T(4)-T(1)*T(3)+T(1)
==> T(1)^3+2*T(1)*T(2)*T(4)-T(2)^2-2*T(4)
==> T(1)^5+4*T(1)^3*T(2)*T(4)+4*T(1)*T(2)^2*T(4)^2
          -T(3)^2+2*T(3)-1
==> T(1)^3*T(2)+2*T(1)*T(2)^2*T(4)-T(1)^2*T(3)
          -2*T(2)*T(3)*T(4)+T(1)^2+2*T(2)*T(4)
==> T(1)^2*T(4)+2*T(2)*T(4)^2
```

with map $A = S/I \to A_1$, $x \mapsto T_1$, $y \mapsto T_2$.

Now the second normalization loop has to be started with A_1. Again the criterion for stopping is not fullfilled, that is, A_1 is not equal to $\mathrm{Hom}_{A_1}(J_1, J_1)$, and the ring A_2 will be computed as affine ring in 4 variables modulo 8 equations. Again, the criterion is not fullfilled, and A_3 is a ring in 3 variables modulo 9 equations. Now in J_3 a zero-divisor of A_3 is found and the ring splits into two rings. Both rings are isomorphic to the polynomial ring in one variable and the algorithm stops with the message:

```
==>// 'normal' created a list of 2 ring(s).
==>// To see the rings, type (if the name of your list
==>// is nor):
==>     show(nor);
```

```
==>// To access the 1-st ring and map (similar for the
==>// others), type:
==>     def R = nor[1]; setring R;  norid; normap;
==>// R/norid is the 1-st ring of the normalization and
==>// normap the map from the original basering to
==>// R/norid
def R1 = nor[1]; setring R1;  norid; normap;
==> norid[1]=0
==> normap[1]=T(1)^2   normap[2]=T(1)^3
def R2 = nor[2]; setring R2;  norid; normap;
==> norid[1]=0
==> normap[1]=-T(1)^3  normap[2]=T(1)^2
```

Hence, $R_1 = \mathbb{Q}[T]$, $R_2 = \mathbb{Q}[T]$, and the normalization of A is given as

$$A \longrightarrow \mathbb{Q}[T] \oplus \mathbb{Q}[T] \qquad x \mapsto (T^2, -T^3) \quad y \mapsto (T^3, T^2)\,.$$

1.5.2. *Non-normal locus.* Let us start with a plane A_k-singularity given by $I = \langle x^2 + y^{k+1} \rangle$ in $\mathbb{Q}[x,y]$, which can easily be done by hand.

We get $I_{\mathrm{Sing}} = \langle x, y^k \rangle$, as test ideal $J = \sqrt{I_{\mathrm{Sing}}} = \langle x, y \rangle$ and $u = x$ as a non-zerodivisor of $S/I = \mathbb{Q}[x,y]/\langle x^2 + y^{k+1} \rangle$. Now

$$(uJ + I) : J \;=\; \langle x^2, xy, y^{k+1} \rangle : \langle x, y \rangle \;=\; \langle x, y^k \rangle\,,$$

hence $I_{NN} = \langle x, x^2 + y^{k+1} \rangle : \langle x, y^k \rangle = \langle x, y \rangle$, as expected. If we do the same with the A_k-surface singularity $I = \langle x^2 + z^2 + y^{k+1} \rangle \subset \mathbb{Q}[x,y,z]$, we get $J = \langle x, y, z \rangle$, $u = z$,

$$(uJ + I) : J = \langle z^2, yz, xz, y^{k+1} + x^2 \rangle : \langle x, y, z \rangle = \langle z, y^{k+1} + x^2 \rangle$$

and, finally, $I_{NN} = \langle u, I \rangle : ((uJ + I) : J) = \langle 1 \rangle$ which is true, since any isolated hypersurface singularity of dimension ≥ 2 is normal.

Let us compute the nonnormal locus of two transversal cusps in the plane, using the procedure `nnlocus` from `normal.lib` in SINGULAR.

```
ring S = 0,(x,y),dp;
ideal I = (x2-y3)*(x3-y2);
ideal NN = nnlocus(I);
```

The radical J of the singular locus is computed as

```
==> J[1]=xy2-y  J[2]=x2y-x  J[3]=y4-x  J[4]=x4-y
```

$u = xy^2 - y$ is chosen as a non-zerodivisor in J and $(uJ + I) : J$ is

```
==> _[1]=xy2-y  _[2]=y4-x2y  _[3]=x3y-y3  _[4]=x4-y
```

Typing `NN;` we get as result the following ideal defining the non-normal locus (equal to J, but with different generators):

```
==> NN[1]=y3-x2  NN[2]=xy2-y  NN[3]=x2y-x  NN[4]=x3-y2
```

1.5.3. *Integral closure of an ideal.* We want to compute the integral closure of the Jacobian ideal I of the E_8-singularity defined by $x^5 + y^3 + z^2 = 0$. Again, we set the printlevel sufficiently high to get intermediate results and comments, which are partly reproduced.

```
LIB "reesclos.lib";
ring A = 0,(x,y,z),dp;
ideal I = jacob(x5+y3+z2);  // the Jacobian ideal
```

Let us first compute the Rees algebra of I.

```
list rees = ReesAlgebra(I);
def Rees = rees[1];
setring Rees;
reesid;
==> reesid[1]=3*y^2*U(3)-2*z*U(2)
==> reesid[2]=5*x^4*U(3)-2*z*U(1)
==> reesid[3]=5*x^4*U(2)-3*y^2*U(1)
def At = rees[2]; setring At;
reesmap;
==> reesmap[1]=x     reesmap[2]=y     reesmap[3]=z
==> reesmap[4]=5x4t  reesmap[5]=3y2t  reesmap[6]=2zt
```

$\mathbb{Q}[x,y,z,U_1,U_2,U_3]$/reesid is isomorphic to the Rees algebra $\mathcal{R}(I)$ as subalgebra of $\mathbb{Q}[x,y,z,t]$, under the map

$$(x,y,z,U_1,U_2,U_3) \longmapsto \big(\text{reesmap}(1),\dots,\text{reesmap}(6)\big)\,.$$

Let us now compute the integral closure of I:

```
list norI = normalI(I);
```

After 3 iterations we reach the normalization of the Rees algebra as $Q[T_1,\dots,T_7]$ modulo the ideal

```
==> 4*T(4)*T(5)-15*T(7)^2
==> 5*T(1)^2*T(7)-2*T(2)*T(4)
==> 3*T(2)^2*T(6)-2*T(3)*T(5)
==> 2*T(1)^2*T(5)-3*T(2)*T(7)
==> 4*T(1)*T(4)*T(5)*T(7)-15*T(1)*T(7)^3
==> T(1)^2*T(2)*T(6)-T(3)*T(7)
==> 5*T(1)^4*T(6)-2*T(3)*T(4)
```

Now we have to determine the map $Q[T_1,\dots,T_7] \to A[t]$. This is computed by representing the ring variables $T_1,\dots,T_7$ as fractions in the variables of the Rees algebra. We get

```
==> T(1) : 25*x^9*z
==> T(2) : 25*x^8*y*z
==> T(2) : 25*x^8*z^2
==> T(2) : 25*x^8*z*U(1)
==> T(2) : 15*x^4*y^2*z*U(1)
==> T(2) : 10*x^4*z^2*U(1)
==> T(2) : 10*x^6*y*z*U(1)
```

with the "universal" denominator: $25x^8z$. Since $\mathcal{R}(I)$ is the image under the map $\mathbb{Q}[x,y,z,U_1,U_2,U_3] \longrightarrow \mathbb{Q}[x,y,z,t]$, $U_1 \mapsto 5x^4t$, $U_2 \mapsto 3y^2t$, $U_3 \mapsto 2zt$, $\mathcal{R}(I)$ is generated in $Q[x,y,z,t]$ by

```
==> generator 1 : x       generator 2 : y
==> generator 3 : z       generator 4 : 5x4t
==> generator 5 : 3y2t    generator 6 : 2zt
==> generator 7 : 2x2yt
```

That is, $\mathcal{R}(I)$ is generated in t-degree 1 and, hence, $\overline{I^k} = \big(\overline{I}\big)^k$ for all $k > 1$. In particular, the integral closure of $I = \langle 5x^4, 3x^2, 2z\rangle$ is generated by 4 elements, the extra element being x^2y. This result is stored in the first entry of the list `norI`:

```
norI[1];
==> _[1]=5x4  _[2]=3y2  _[3]=2z  _[4]=2x2y
```

2. Effective Construction of Algebraic Geometry Codes

Goppa's construction of linear codes using algebraic geometry, the so-called *geometric Goppa* or *AG codes*, was a major breakthrough in the history of coding theory. In particular, it was the first (and only) construction leading to a family of codes with parameters above the Gilbert-Varshamov bound [41].

There exist several (essentially equivalent) ways to construct AG codes starting from a smooth projective curve $\widetilde{C}$ defined over a finite field $\mathbb{F}$. Mainly, we should like to mention the L-, resp. the Ω-*construction*.

Given rational points $Q_1, \dots, Q_n \in \widetilde{C}$ and a rational divisor G on $\widetilde{C}$ having disjoint support with the divisor $D = Q_1 + \ldots + Q_n$, the AG code $C_L(G, D, \widetilde{C})$, resp. $C_\Omega(G, D, \widetilde{C})$, is the image of the $\mathbb{F}$-linear map

$$\begin{aligned} ev_D : \mathcal{L}(G) \longrightarrow \mathbb{F}^n, &\quad f \longmapsto \big(f(Q_1), \dots, f(Q_n)\big)\,, \text{ resp.} \\ res_D : \Omega(G-D) \longrightarrow \mathbb{F}^n, &\quad \omega \longmapsto \big(\mathrm{res}_{Q_1}(\omega), \dots, \mathrm{res}_{Q_n}(\omega)\big)\,. \end{aligned}$$

In practice, there are two main difficulties when looking for an *effective method* to compute the generator matrices of such codes: Given a plane (singular) model C of $\widetilde{C}$, how to compute the places of C and how to compute a basis for the linear system $\mathcal{L}(G)$ (cf. below), resp. for the vector space of rational one-forms

$$\Omega(G-D) = \big\{\omega \in \Omega(X)^* \,\big|\, (\omega) + G - D \geq 0\big\} \cup \{0\}\,.$$

One possible solution, making use of blowing-up theory (to compute the places of C) and of the (classical) Brill-Noether algorithm (for the computation of a basis of $\mathcal{L}(G)$) is presented in [28, 19]. In the following, we should like to point to the modified approach of Campillo and Farrán [3], using Hamburger-Noether expansions instead of blowing-up theory, and to present in some detail the resulting algorithm as implemented in the computer algebra system Singular [17].

2.1. **Preliminaries, Notations.**

Throughout the following, let $\mathbb{F}$ be a finite field and $\overline{\mathbb{F}}$ an algebraic closure of $\mathbb{F}$. Moreover, let $C \subset \mathbb{P}^2(\mathbb{F})$ be an *absolutely irreducible*, reduced, projective plane curve given by a homogeneous form of degree d, $F \in \mathbb{F}[X, Y, Z]_d$.

A point $P \in C$ is called *rational* if its coordinates are in $\mathbb{F}$. More generally, by a *closed point* $[P] \in C$ we denote the formal sum of a point (defined over $\overline{\mathbb{F}}$) with its conjugates. If there is no risk of confusion, we sometimes write $P \in C$ to denote the closed point $[P]$. Note that closed points are invariant under the action of the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$.

We denote by $n : \widetilde{C} \to C$ the normalization and by

$$N := i \circ n : \widetilde{C} \to \mathbb{P}^2(\mathbb{F})$$

the parametrization of C (with $i : C \hookrightarrow \mathbb{P}^2(\mathbb{F})$ being the inclusion).

The points of $\widetilde{C}$ are called *places of* C. Again, a place is called *rational* if its coordinates are in $\mathbb{F}$, and by a *closed place* we denote the formal sum of a place (defined over $\overline{\mathbb{F}}$) with its conjugates. Note that each smooth (rational, resp. closed) point $P \in C$ corresponds to a unique (rational, resp. closed) place $n^{-1}(P) \in \widetilde{C}$. If P is a singular point of C then each local branch of C at P corresponds to a unique place of C. Hence, the set of places of C can be identified with the set consisting of the non-singular points of C and all tuples $\big(P;(C_i,P)\big)$, P a singular point of C and (C_i,P) a local branch of C at P.

A *(rational) divisor* D on $\widetilde{C}$ is a finite, weighted, formal sum of (closed) places Q of C, $D=\sum n_Q Q$ with integer coefficients $n_Q =: \mathrm{ord}_Q(D)$. The divisor D is called *effective* if there are no negative n_Q. Moreover, we introduce the *degree* of the divisor D, $\deg D := \sum n_Q$ and the *support* of D, $\mathrm{supp}\, D := \{Q \mid n_Q \neq 0\}$.

To each element g in the function field $\mathbb{F}(\widetilde{C})$ of $\widetilde{C}$ one associates the *principal divisor* $(g) := \sum \mathrm{ord}_Q(g)\cdot Q$. Note that (g) has degree 0, by the residue theorem.

Finally, the *linear system* associated to a divisor D is defined to be

$$\mathcal{L}(D) \;:=\; \big\{\, g\in \mathbb{F}(\widetilde{C}) \;\mid\; (g) \geq -D \,\big\} \cup \{\, 0 \,\}\,,$$

where the order relation is defined by

$$D \geq D' \;:\Longleftrightarrow\; \mathrm{ord}_Q(D) \geq \mathrm{ord}_Q(D') \text{ for all places } Q \in \widetilde{C}\,.$$

2.2. **Symbolic Hamburger-Noether expressions.** We recall the definition of Hamburger-Noether expansions (HNE), resp. symbolic Hamburger-Noether expressions, for the branches of a reduced plane curve singularity. They can be regarded as the analogue of Puiseux expansions when working over a field of positive characteristic (cf. the discussion in [2], 2.1). As being the case for Puiseux expansions, many invariants of plane curve singularities (such as the multiplicity sequence, the δ-invariant, the intersection multiplicities of the branches, etc.), can be computed directly from the corresponding system of HNE.

Let $P\in C$ be a point and x,y local parameters at P. Moreover, let the germ (C,P) be given by a local equation $f\in \mathbb{F}[[x,y]]$ with irreducible decomposition $f=f_1\cdot\ldots\cdot f_r \in \overline{\mathbb{F}}[[x,y]]$. Finally, let's suppose that x is a *transversal parameter* for (C,P), that is, the order of $f(0,y)$ is equal to the order of $f(x,y)$.

Definition 2.1. A *Hamburger-Noether expansion (HNE)* of C at P for the local branch given by f_ν (defined over some finite algebraic extension $\mathbb{F}\subset\mathbb{F}_{\mathrm{ext}}$) is a finite sequence of equations

$$\begin{array}{rcl} z_{-1} &=& a_{0,1}z_0 + a_{0,2}z_0^2 + \ldots + a_{0,h_0}z_0^{h_0} + z_0^{h_0}z_1 \\ z_0 &=& \phantom{a_{0,1}z_0 + {}} a_{1,2}z_1^2 + \ldots + a_{1,h_1}z_1^{h_1} + z_1^{h_1}z_2 \\ \vdots && \qquad\qquad \vdots \\ z_{i-1} &=& \phantom{a_{0,1}z_0 + {}} a_{i,2}z_i^2 + \ldots + a_{i,h_i}z_i^{h_i} + z_i^{h_i}z_{i+1} \\ \vdots && \qquad\qquad \vdots \\ z_{s-2} &=& \phantom{a_{0,1}z_0 + {}} a_{s-1,2}z_{s-1}^2 + \ldots + a_{s-1,h_{s-1}}z_{s-1}^{h_{s-1}} + z_{s-1}^{h_{s-1}}z_s \\ z_{s-1} &=& \phantom{a_{0,1}z_0 + {}} a_{s,2}z_s^2 + a_{s,3}z_s^3 + \ldots\ldots\ldots\ldots \end{array} \tag{1}$$

where s is a non-negative integer, $a_{j,i}\in\mathbb{F}_{\mathrm{ext}}$ and h_j, $j=1,\ldots,s-1$, are positive integers, such that $f_\nu\big(z_0(t),z_{-1}(t)\big)=0$ in $\mathbb{F}_{\mathrm{ext}}[[z_s]]$.

If the local equation of (C,P) is polynomial in x,y, i.e., $f\in\mathbb{F}[x,y]$, then the last (infinite) row of (1) can be replaced, equivalently, by a (finite) implicit equation

$$g(z_s,z_{s-1}) \;=\; 0\,, \qquad g\in\mathbb{F}_{\mathrm{ext}}[x,y]\,, \quad \frac{\partial g}{\partial z_{s-1}}(0,0)\neq 0\,.$$

The resulting system is called a *symbolic Hamburger-Noether expression* (sHNE) for the branch.

Any HNE leads to a primitive parametrization $\varphi : \mathbb{F}[[x,y]] \to \mathbb{F}_{\mathrm{ext}}[[t]]$ of the branch (setting $t := z_s$ and mapping $x \mapsto z_0(z_s)$, $y \mapsto z_{-1}(z_s)$). It can be computed from a sHNE up to an arbitrary finite degree in t.

Remark 2.2. There exist constructive algorithms to compute a system of sHNE's (resp. HNE's up to a given degree) for the branches of a reduced plane curve singularity (cf. [2] for the irreducible case, resp. [34] for the reducible case). A modification of the latter algorithm is implemented in the computer algebra system Singular [26, 17].

To perform the algorithm one does not need any knowledge about the irreducible factorization of f in $\overline{\mathbb{F}}[[x,y]]$. Moreover, in the reducible case, the probably necessary (finite) field extension is not performed a priori, but by successive (primitive) extensions introduced exactly when needed for computations (factorization).

Remark 2.3. Given a Hamburger-Noether expansion (1) for a local branch (C_ν, P), the corresponding *multiplicity sequence* $m_1, \dots, m_n$ can be easily read off:

$$\underbrace{n_0,\dots,n_0}_{h_0 \text{ times}}, \underbrace{n_1,\dots,n_1}_{h_1 \text{ times}}, \dots\dots, \underbrace{n_{s-1},\dots,n_{s-1}}_{h_{s-1} \text{ times}}, 1, \dots, 1$$

where the n_j are defined recursively by setting $n_s := 1$ and, for $j = s, \dots, 1$,

$$n_{j-1} := \begin{cases} n_j h_j + n_{j+1} & \text{if } a_{j,\ell} = 0 \text{ for all } 2 \le \ell \le h_j, \\ n_j \ell & \text{otherwise, with } \ell \ge 2 \text{ minimal s.th. } a_{j,\ell} \ne 0. \end{cases}$$

In particular, we can compute the δ*-invariant* of the branch directly from the Hamburger-Noether expansion, since

$$\delta(C_\nu, P) = \sum_{i=1}^{n} \frac{m_i(m_i-1)}{2}.$$

Moreover, if $\varphi : \mathbb{F}[[x,y]] \to \mathbb{F}_{\mathrm{ext}}[[t]]$ is the primitive parametrization of (C_ν, P) obtained from the HNE then the intersection multiplicity of (C_ν, P) with the plane curve germ (C', P) given by $g \in \overline{\mathbb{F}}[[x,y]]$ can be computed as $i_P(C_\nu, C') = \mathrm{ord}_t\, g\big(\varphi(x), \varphi(y)\big)$.

2.3. **Adjoint curves and Brill-Noether residue theorem.** This section is devoted to the "core" of the presented algorithm, the Brill-Noether residue theorem. In an intuitive formulation it states that "the intersection divisors of C with all adjoint curves of a fixed degree m form a complete linear system, up to being shifted by the adjoint divisor".

Let P be a closed point of C. We introduce the local (resp. semilocal) rings $\mathcal{O} := \mathcal{O}_{C,P}$, resp. $\overline{\mathcal{O}} := (n_* \mathcal{O}_{\widetilde{C}})_P \cong \prod_{i=1}^r \mathcal{O}_{\widetilde{C},Q_i}$, where $Q_1, \dots, Q_r$ denote the closed places of C over P. Note that $\mathcal{O}$ is a subring of $\overline{\mathcal{O}}$. The *conductor*

$$\mathcal{C}_P := \mathcal{C}_{\overline{\mathcal{O}}/\mathcal{O}} := \big\{ \phi \in \overline{\mathcal{O}} \mid \phi \cdot \overline{\mathcal{O}} \subset \mathcal{O} \big\}$$

is an ideal both in $\mathcal{O}$ and in $\overline{\mathcal{O}}$. It defines a divisor $\mathcal{A}_P$ on $\widetilde{C}$ whose support is $\{Q_1, \dots, Q_r\}$, the set of all places of C over the singular point P. Recall that such a place corresponds to a unique branch of (C, P). By abuse of notation, we sometimes do not distinguish between the place and the corresponding branch.

Definition 2.4. We call the divisor $\mathcal{A} := \sum_P \mathcal{A}_P$ the *adjunction divisor* (or the *divisor of double points*) of C. Its support is the set of all places over singular points of C.

Remark 2.5. The adjunction divisor $\mathcal{A} = \sum_Q d_Q Q$ is rational, that is, conjugate branches Q, Q' satisfy $d_Q = d_{Q'}$. We have $d_Q = 2 \cdot \delta(C,P)$ if Q is a place over an *irreducible* plane curve singularity (C,P), resp.

$$d_Q \;=\; 2 \cdot \delta(C_i,P) + \sum_{j \neq i} i_P(C_i,C_j) \tag{2}$$

if Q corresponds to the local branch (C_i,P) of (C,P) (δ denoting the δ-invariant and i_P the local intersection multiplicity at P). Alternatively, one can use the *Dedekind formula* to compute the multiplicities d_Q: let $\varphi_Q : \mathbb{F}[[x,y]] \to \overline{\mathbb{F}}[[t]]$ be a primitive parametrization of the branch Q then

$$d_Q \;=\; \operatorname{ord}_t \left(\frac{f_y\big(\varphi_Q(x), \varphi_Q(y)\big)}{\frac{d}{dt}\varphi_Q(x)} \right) \;=\; \operatorname{ord}_t \left(\frac{f_x\big(\varphi_Q(x), \varphi_Q(y)\big)}{\frac{d}{dt}\varphi_Q(y)} \right)$$

if the respective expressions are finite (notice that either $\frac{d}{dt}\varphi_Q(x)$ or $\frac{d}{dt}\varphi_Q(y)$ does not vanish identically).

Notation. Let $H \in \mathbb{F}[X,Y,Z]_m$ be a homogeneous form of degree m such that F does not divide H. Then we denote by N^*H the intersection divisor on $\widetilde{C}$ cut out by the (preimage under N of the) plane curve defined by H.

Definition 2.6. Let D be a rational divisor on $\mathbb{P}^2(\mathbb{F})$ such that C is not contained in the support of D. We call D an *adjoint divisor* of C iff the pull-back divisor satisfies $N^*D \geq \mathcal{A}$.

Let $H \in \mathbb{F}[X,Y,Z]_m$, $F \not| H$, such that $N^*H \geq \mathcal{A}$. Then we call H an *adjoint form* and the plane curve defined by H an *adjoint curve* of C.

Note that D is an adjoint divisor iff the intersection multiplicity of D with every local branch Q of C is at least d_Q.

Proposition 2.7 (Brill-Noether residue theorem)**.** *Let $C \overset{i}{\hookrightarrow} \mathbb{P}^2(\mathbb{F})$ be a reduced absolutely irreducible plane projective curve given by the homogeneous polynomial $F \in \mathbb{F}[X,Y,Z]_d$. Moreover, let $n : \widetilde{C} \to C$ be the normalization and D a rational divisor on $\widetilde{C}$.*

Finally, let $H_0 \in \mathbb{F}[X,Y,Z]_m$ be an adjoint form of degree $m > 0$ such that $N^*H_0 \geq \mathcal{A} + D$. Then we can identify

$$\mathcal{L}(D) \;\equiv\; \left\{ \left[\frac{H}{H_0}\right] \in \mathbb{F}(C) \;\middle|\; \begin{array}{c} H \in \mathbb{F}[X,Y,Z]_m,\; F \not| H, \\ N^*H \geq N^*H_0 - D \end{array} \right\} \cup \{0\}$$

under the isomorphism $\mathbb{F}(\widetilde{C}) \cong \mathbb{F}(C)$ induced by N.

This proposition is an immediate corollary of

Theorem 2.8 (M. Noether)**.** *Let $G, H \in \mathbb{F}[X,Y,Z]$ be homogeneous forms such that $N^*H \geq \mathcal{A} + N^*G$. Then there exist homogeneous forms $A, B \in \mathbb{F}[X,Y,Z]$ such that $H = AF + BG$.*

For a complete proof we refer to [42], pp. 215ff, resp. [28], Prop. 4.1.

Remark 2.9. Haché [18] has shown that a form $H_0 \in \mathbb{F}[X,Y,Z]_m$ as in the Brill-Noether residue theorem exists whenever

$$m \;>\; \max\left\{ d-1\,,\; \frac{d-3}{2} + \frac{\deg\left(\mathcal{A} + D_+\right)}{d} \right\}, \tag{3}$$

where D_+ denotes the effective part of the divisor $D = D_+ - D_-$.

2.4. **Computational Aspects.** Let C be an absolutely irreducible plane projective curve defined over the prime field $\mathbb{F} = \mathbb{F}_p$ (given by a homogeneous form $F \in \mathbb{F}[X,Y,Z]_d$).

2.4.1. *Computing the places of* C. A place Q of C is represented by a triple, consisting of

- the closed point $[P] \in C$ corresponding to Q,
- the *degree* k_Q of the place (that is, the minimal degree of a field extension defining Q) and
- a symbolic Hamburger-Noether expression *HN* for the local branch corresponding to Q (defined over a primitive algebraic field extension $\mathbb{F} \subset \mathbb{F}(a)$ of degree k_Q).

Recall that $[P] \in C$ denotes the formal sum of a point $P \in C$ with its conjugates. Affine closed points will be represented by a defining (triangular) ideal $I = \langle \phi, \psi \rangle \subset \mathbb{F}[x,y]$, while closed points at infinity are usually stored in form of a homogeneous polynomial $\Phi \in \mathbb{F}[X,Y]$ (the defining prime factor of $F(X,Y,0)$).

Note that the conjugates of a place Q are given by the triples $([P], k_Q, HN')$, where HN' runs through the conjugates of *HN*. Hence, when computing the closed places of C, we can restrict ourselves to computing one representing place for each. We apply the following algorithm:

Input: Squarefree homogeneous polynomial $F \in \mathbb{F}[X,Y,Z]_d$, degree bound $k \in \mathbb{N}$.

Output: List L of all closed singular places and all closed non-singular places up to degree k of the plane curve C defined by F.

1. *Affine singular points.* Let $f(x,y) := F(x,y,1)$ and $I := \langle f, f_x, f_y \rangle$ the Tjurina ideal of f. Compute a *triangular system* for I, that is, a system of triangular bases T_i such that $V(I) = V(T_1) \cup \ldots \cup V(T_s)$.

   Here, by a *triangular basis* one denotes a reduced lexicographical Gröbner basis of the form $T = \{\phi, \psi\}$ with $\phi \in \mathbb{F}[y]$ a monic polynomial in y and $\psi = y^b + \sum_{i<b} \psi_i(x) y^i \in \mathbb{F}[x,y]$. Triangular systems can be computed effectively, basically by two different methods, one due to Lazard [27, 7], the other due to Möller [31]. Choose any of these methods to compute a triangular system for I, $T_i = \{\phi_i, \psi_i\}$, $i = 1, \ldots, s$. For each i,
   - compute a prime factorization of ϕ_i in $\mathbb{F}[y]$,
   $$\phi_i = \phi_{i,1} \cdot \ldots \cdot \phi_{i,r_i} \in \mathbb{F}[y]\,;$$
   - for $j = 1, \ldots, r_i$, let $\mathbb{F}(a_{i,j}) = \mathbb{F}[y]/\langle \phi_{i,j} \rangle$ be the primitive field extension defined by the irreducible polynomial $\phi_{i,j}$. Compute a prime factorization of ψ_i in $\mathbb{F}(a_{i,j})[x,y]$,
   $$\psi_i = \psi_{i,1} \cdot \ldots \cdot \psi_{i,s_i} \in \mathbb{F}(a_{i,j})[x,y]\,.$$

   Finally, the closed affine singular points are given by the set of ideals
   $$\mathrm{SING_{aff}} := \left\{ \langle \phi_{i,j}, \overline{\psi}_{i,k} \rangle \subset \mathbb{F}[x,y] \;\middle|\; \begin{array}{c} j = 1, \ldots, r_i\,,\ k = 1, \ldots, s_i\,, \\ i = 1, \ldots, s \end{array} \right\},$$
   where $\overline{\psi}_{i,j}$ is the image of $\psi_{i,j}$ when substituting the parameter $a_{i,j}$ by y.

2. *Points at infinity.* Let $f_\infty(x) := F(x,1,0)$ and compute a prime factorization of the polynomial $f_\infty \in \mathbb{F}[x]$,

$$f_\infty = f_{\infty,1} \cdot \ldots \cdot f_{\infty,d'} \in \mathbb{F}[x]\,, \qquad d' \leq d\,. \tag{4}$$

Let $a_j \in \overline{\mathbb{F}}$ be a root of $f_{\infty,j}$ and define

$$\mathrm{PTS}_\infty := \left\{ \big[(a_j:1:0)\big] \;\middle|\; j = 1,\ldots,d' \right\},$$

where $\big[(a_j:1:0)\big]$ denotes the formal sum of the point $(a_j:1:0)$ (defined over $\overline{\mathbb{F}}$) with its conjugates. (It is represented by $f_{\infty,j}$.)

We denote by $\mathrm{SING}_\infty \subset \mathrm{PTS}_\infty$ the subset of closed singular points. To check whether a point $\big[(a_j:1:0)\big]$ is singular or not, one has to check whether $F_X(a_j,1,0) = F_Z(a_j,1,0) = 0$ (these computations can be performed over the finite field extension $F(a_j) = F[x]/\langle f_{\infty,j}\rangle$).

Finally, consider the (closed) point $P = (1:0:0)$: if $F(1,0,0) = 0$ then P has to be added to PTS_∞; if, additionally, $F_Y(1,0,0)$ and $F_Z(1,0,0)$ vanish then it has to be added to SING_∞, too.

The sets PTS_∞ (resp. SING_∞) are the sets of closed (singular) points at infinity.

3. *Affine singular places.* To each closed affine singular point $[P]$ given by a (triangular) ideal $\langle \phi, \psi\rangle \in \mathrm{SING}_{\mathrm{aff}}$ we compute the corresponding places in form of a system of symbolic Hamburger-Noether expressions for the respective germ (C,P) (defined over $\overline{\mathbb{F}}$). More precisely, a closed place over $[P]$ is the formal sum of a place Q described by one of the computed sHNE with its conjugates.

   The computation of the symbolic Hamburger-Noether expressions has to be performed in the local ring $\mathbb{F}(a)[x,y]_P$ where $\mathbb{F} \subset \mathbb{F}(a)$ is a primitive field extension (of degree $k_P = \deg_y(\phi)\deg_x(\psi)$) such that ϕ,ψ decompose into linear factors. Note that during the computation of a sHNE further field extensions might be necessary.

4. *Singular places at infinity.* To each closed singular point $[P]$ in SING_∞ we compute a system of sHNE for the local germ (C,P) (defined over $\overline{\mathbb{F}}$). To be precise, if $P = (a_j:1:0)$ then we compute a system of sHNE for $F(x+a_j,1,z)$ in $\mathbb{F}(a_j)[x,z]_{\langle x,z\rangle}$; if $P = (1:0:0)$ then the system of sHNE is computed for $F(1,y,z)$ in $\mathbb{F}[y,z]_{\langle y,z\rangle}$.

5. *Non-singular affine closed points up to degree k.* For each $1 \leq \ell \leq k$ do the following:
   - let $Q := p^\ell$ and set $I := \langle f,\, x^Q - x,\, y^Q - y\rangle$.
   - Proceed as in Step 1 to obtain a set of (triangular) ideals $\mathrm{PTS}_{\mathrm{aff}}$ corresponding to the set of closed points defined over $\mathbb{F}_Q$.
   - For all non-singular $[P] \in \mathrm{PTS}_{\mathrm{aff}}$ (given by $\langle\phi,\psi\rangle \subset \mathbb{F}[x,y]$) compute the degree $k_P = \deg_y(\phi)\deg_x(\psi)$. If $k_P = \ell$ then compute the corresponding closed place (that is, a sHNE for the germ (C,P)) and add it to the list L of closed places.

Remark 2.10. It is interesting to notice that triangular sets have mainly been used for numerical purpose, since they allow a fast and stable numerical solving of polynomial systems (cf. [27, 31, 13]), and this has been the reason for implementing it in SINGULAR. Several experiments have shown that they behave also superior against other methods to represent closed points over finite fields.

2.4.2. *Computing the adjunction divisor.* For any (closed) singular place we determine the multiplicity $d_{[Q]} := \mathrm{ord}_Q(\mathcal{A})$, alternatively, by the formula (2) or the Dedekind formula (cf. Remarks 2.5 and 2.3). Note that to compute a local intersection number $i_P(f,g)$ (as appearing in both formulas), we can proceed inductively, computing the primitive parametrization $\varphi : K[[x,y]] \to K[[t]]$ of g up to degree k, until $\mathrm{ord}_t(\varphi(x) \bmod t^k, \varphi(y) \bmod t^k)$ is less than $k/\mathrm{ord}(f)$.

2.4.3. *Computing the divisors N^*H.* The intersection divisors N^*H can be computed by the following algorithm:

Input: $F \in \mathbb{F}[X,Y,Z]_d$, L list of closed places $[Q]$ of C (defined by $F=0$), $H \in \mathbb{F}[X,Y,Z]_m$.

Output: Extended list of closed places $[Q]$ and list of integers $m_{[Q]}$ such that $N^*H = \sum_{[Q]} m_{[Q]}[Q]$.

1. *Affine Intersection.* Let $h(x,y) := H(x,y,1)$, $f(x,y) := F(x,y,1)$ and consider $I := \langle h, f\rangle$. Proceed as in Step 1 of the algorithm in Section 2.4.1 to obtain a set of (triangular) ideals corresponding to the set of closed points in $V(I)$. For each closed point $[P]$ in $V(I)$ do the following:
   - check for each closed place $[Q]$ in L whether $[Q]$ lies above $[P]$ (that is, the first entry of the triple representing $[Q]$ has to be $[P]$). If this is the case, compute the multiplicity $m_{[Q]} = \mathrm{ord}_t\, h\big(\varphi_Q(x), \varphi_Q(y)\big)$, where φ_Q is the primitive parametrization obtained from the sHNE (third entry) of $[Q]$.
   - If there is no such $[Q]$ in L then compute the sHNE for the (smooth) germ (C,P), add the resulting place $[Q]$ to the list L and proceed as before to obtain $m_{[Q]}$.
2. *Intersection at infinity.* Let $h_\infty(x) := H(x,1,0)$ and compute a prime factorization of the polynomial $h_\infty \in \mathbb{F}[x]$,
$$h_\infty = h_{\infty,1} \cdot \ldots \cdot h_{\infty,d''} \in \mathbb{F}[x]\,, \qquad d'' \leq d\,.$$
   Each factor that appeared also in the prime factorization (4) of f_∞ corresponds to a closed point $\big[(a_j:1:0)\big]$ in the intersection of C with the plane curve defined by H. For the corresponding closed places $[Q]$ we compute
$$m_{[Q]} = \mathrm{ord}_t\, H\big(\varphi_Q(x)+a_j,\, 1,\, \varphi_Q(z)\big)\,,$$
   where $\varphi_Q : \mathbb{F}[[x,z]] \to \overline{\mathbb{F}}[[t]]$ is the primitive parametrization obtained from the sHNE of Q (cf. Step 4 of the algorithm in Section 2.4.1).

   Finally, if $(1:0:0) \in \mathrm{PTS}_\infty$ and $H(1,0,0)=0$ then we compute for the corresponding closed places $[Q]$ the multiplicities
$$m_{[Q]} = \mathrm{ord}_t\, H\big(1, \varphi_Q(y), \varphi_Q(z)\big)\,.$$

2.4.4. *Computing $\mathbb{F}$-bases of adjoint forms.* Let $\mathrm{Monom}(m) = \{G_1,\ldots,G_M\}$ denote the monomial $\mathbb{F}$-basis for $\mathbb{F}[X,Y,Z]_m$, resp. the corresponding vector of monomials. We represent a homogeneous form $H \in \mathbb{F}[X,Y,Z]_m$ by the vector $v \in \mathbb{F}^n$ such that $H = \mathrm{Monom}(m)^T \cdot v$. To compute $\mathbb{F}$-bases of adjoint forms as needed by the Brill-Noether algorithm, we apply the following algorithm:

Input: $F \in \mathbb{F}[X,Y,Z]_d$, L list of closed places of C (defined by $F=0$), $m > d$ a positive integer,
non-negative integers $d_{[Q]}$, $[Q] \in L$, s.th. $\mathcal{A} = \sum_{[Q]} d_{[Q]}[Q]$,
non-negative integers $n_{[Q]}$, $[Q] \in L$, s.th. $E = \sum_{[Q]} n_{[Q]}[Q]$.

Output: $\mathbb{F}$-basis of a subspace of

$$V := \left\{ H \in \mathbb{F}[X,Y,Z]_m \;\middle|\; F \mid H \text{ or } N^*H \geq \mathcal{A} + E \right\}$$

complementary to $W := F \cdot \mathbb{F}[X,Y,Z]_{m-d}$ (given in form of a matrix of coefficients w.r.t. $\mathrm{Monom}(m)$).

1. *The subspace $W \subset V$ of forms divisible by F.* Compute $W \in \mathrm{Mat}(\ell \times M, \mathbb{F})$ with $W \cdot \mathrm{Monom}(m) = F \cdot \mathrm{Monom}(m-d)$.
   Note that $\ell = \binom{m-d+2}{2}$, $M = \binom{m+2}{2}$.
2. *The space V.* For each closed place $[Q]$ with $m_{[Q]} := d_{[Q]} + n_{[Q]} > 0$ do the following:
   - compute a matrix $A_Q = (a_{ij})_{i,j} \in \mathrm{Mat}(m_{[Q]} \times M, \overline{\mathbb{F}})$ such that

   $$G_j\big(\varphi_Q(x), \varphi_Q(y)\big) = \sum_{i=0}^{m_{[Q]}-1} a_{ij} t^i\,, \quad j = 1, \ldots, M\,,$$

     where φ_Q is the primitive parametrization as computed from the sHNE of Q (up to degree $m_{[Q]}$).
   - Let k_Q be the degree of the place Q and let $A_Q^{(i)}$ denote the image of A_Q after applying i times the Frobenius map over $\mathbb{F}$. Then compute

   $$A_{[Q]} \;:=\; \text{row-red NF} \begin{pmatrix} A_Q \\ A_Q^{(1)} \\ \vdots \\ A_Q^{(k_Q-1)} \end{pmatrix} \in \mathrm{Mat}(k_Q m_{[Q]} \times M, \mathbb{F})\,.$$

   Finally, concatenate the $A_{[Q]}$ to obtain a matrix $A \in \mathrm{Mat}(K \times M, \mathbb{F})$ with $K = \sum_{[Q]} k_Q m_{[Q]}$ and compute V as the kernel of A, that is

   $$V := \mathrm{Ker}(A) \in \mathrm{Mat}(k \times M, \mathbb{F})\,, \qquad A \cdot V^T = 0\,.$$
3. *Compute a complement of W in V.* This can be done, for instance, by using the `lift` command in SINGULAR.

2.4.5. *Computing a basis for $\mathcal{L}(G)$.* Let G be any rational divisor on $\widetilde{C}$. The algorithms of the preceding sections finally allow to compute an $\mathbb{F}$-basis of $\mathcal{L}(G)$ by the *Brill-Noether Algorithm* (cf. Prop. 2.7):

Input: $F \in \mathbb{F}[X,Y,Z]_d$, L list of closed places of $C = \{F=0\}$,
non-negative integers $d_{[Q]}$, $[Q] \in L$, s.th. $\mathcal{A} = \sum_{[Q]} d_{[Q]}[Q]$,
integers $g_{[Q]}$, $[Q] \in L$, s.th. $G = \sum_{[Q]} g_{[Q]}[Q]$.

Output: Vector space basis of $\mathcal{L}(G)$ (in terms of rational functions on C).

1. *Choose m sufficiently large* (for instance, according to (3), above).
2. *Compute $H_0 \in \mathbb{F}[X,Y,Z]_m$, $F \nmid H_0$, such that $N^*H_0 \geq \mathcal{A} + G_+$.* To do so, we compute an $\mathbb{F}$-basis for a vector subspace of

$$V_0 := \left\{ H \in \mathbb{F}[X,Y,Z]_m \;\middle|\; F \mid H \text{ or } N^*H \geq \mathcal{A} + G_+ \right\}\,,$$

complementary to $W := F \cdot \mathbb{F}[X,Y,Z]_{m-d}$ (cf. Section 2.4.4) and choose any element H_0 of this basis (for instance, with the minimal number of monomials).

3. *Compute the effective divisor* $R := N^*H_0 - G - \mathcal{A}$ (cf. Section 2.4.3).
4. *Compute an* $\mathbb{F}$*-basis* $H_1, \ldots, H_s$ *of a vector subspace of*
$$V := \{H \in \mathbb{F}[X,Y,Z]_m \mid F \mid H \text{ or } N^*H \geq \mathcal{A} + R\},$$
complementary to $W = F \cdot \mathbb{F}[X,Y,Z]_{m-d}$ (cf. Section 2.4.4).
5. *Return the set of rational functions* $\mathcal{B} := \left\{ \dfrac{H_1}{H_0}, \ldots, \dfrac{H_s}{H_0} \right\}$.

2.5. **Example.** The above algorithms are implemented in the library `brnoeth.lib` of Singular, together with procedures for coding and decoding and are distributed with version 2.0. To compute an example, we first have to load the library.

```
LIB "brnoeth.lib";
```

Let C be the absolutely irreducible plane projective curve given by the affine equation $y^2 + y + x^5 \in \mathbb{F}_2[x,y]$. We compute all places up to degree 4, by

```
ring r=2,(x,y),dp;
poly f=y2+y+x5;
list CURVE=Adj_div(f);
==> The genus of the curve is 2.
CURVE=NSplaces(3,CURVE); // places up to degree 4=1+3
```

We can consider the curve as being defined over $\mathbb{F}_{16}$ in order to get many *rational* places.

```
CURVE=extcurve(4,CURVE);
==> Total number of rational places : NrRatPl = 33
```

The degree of the computed (conjugacy classes of) places is displayed by

```
list L=CURVE[3]; L;
==>  [1]: 1,1   [2]: 1,2   [3]: 1,3
==>  [4]: 2,1
==>  [5]: 3,1   [6]: 3,2
==>  [7]: 4,1   [8]: 4,2   [9]: 4,3  [10]: 4,4
==> [11]: 4,5  [12]: 4,6  [13]: 4,7
```

In particular, besides the $33 = 3*1 + 1*2 + 7*4$ rational places over $\mathbb{F}_{16}$ there are 2 closed places $[Q_5], [Q_6]$ of degree 3. The adjunction divisor is given by $8Q_1$, where Q_1 is the unique (rational) point on $\widetilde{C}$ mapping to the singular point $(0:1:0)$. This can be read off as follows:

```
CURVE[4];     // the mult's d_Q at L[1],L[2],...
              // (zeroes omitted)
==> 8
def r1=CURVE[5][1][1];
setring r1;
POINTS[1];   // coordinates of the base point of L[1]
==> [1]:  0  [2]:  1   [3]:  0
PARAMETRIZATIONS[1];  // parametrization of L[1]
==> [1]:  _[1]=t3+t8
          _[2]=t5+t15
    // exact up to order:
```

```
[2]:  8,10
```

We construct the evaluating AG-code $C_L(G, D, \widetilde{C})$ where all rational points of $\widetilde{C}$ appear in the support of D and $G = [Q_5] + [Q_6]$.

```
intvec G=0,0,0,0,1,1,0,0,0,0,0,0,0;
intvec D=1..33;
def R=CURVE[1][4];
setring R;
matrix CODE=AGcode_L(G,D,CURVE);
```

The echelon form of the resulting 5×33-matrix is

$$\left(\begin{array}{cccccccccccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & a & a^8 & a^{12} & a^{12} & 0 & a & a^8 & a & a^5 & a^5 & a^9 & a^5 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & a^8 & a^6 & a^7 & a^5 & a^{10} & a^4 & 1 & a^{12} & a^2 & a^2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & a^7 & 0 & a^{13} & a^9 & a^5 & a^6 & a^5 & a^{11} & a^{11} & a^{14} & a^{14} & a^6 \\ 0 & 0 & 0 & 1 & 1 & 0 & a^4 & a^{13} & a^6 & a^6 & 0 & a^4 & a^{13} & a^{13} & a^{13} & a^5 & a^8 & a^5 \\ 0 & 0 & 0 & 0 & 0 & 1 & a^7 & a^6 & a^4 & a^4 & 1 & a^7 & a^6 & 1 & a^7 & a^6 & a^4 & a^6 \end{array}\right.$$

$$\left.\begin{array}{ccccccccccccccc} a^9 & a & a^5 & a^7 & a & a^7 & a & a^3 & a^{13} & a^4 & a^{12} & a^4 & a^{12} & a^3 & a^{13} \\ a^5 & a^8 & a^6 & a^4 & a^{14} & a^2 & a^{12} & a^4 & a^4 & a^{11} & a^5 & a^8 & a^{12} & a^{11} & a^{13} \\ a^7 & a^9 & a^{13} & a^{14} & a^6 & a^{11} & a^9 & a^5 & a^{13} & a^5 & a^7 & a^{13} & a & a^7 & a^4 \\ a^8 & a^{13} & a^{13} & a^3 & a^7 & a^3 & a^7 & a^{10} & a^{11} & a^{14} & a & a^{14} & a & a^{10} & a^{11} \\ a^4 & 1 & a^7 & a^3 & a^{13} & a^3 & a^{13} & a^7 & a^6 & a^4 & 1 & a^4 & 1 & a^7 & a^6 \end{array}\right).$$

Note that the constructed code $C_L(G, D, \widetilde{C})$ has block length 33, dimension 5 and designed distance $27 = 33 - \deg G$. On the other hand, the first row corresponds to a word of weight 27, whence the designed distance coincides with the minimal distance. As a result, we get that $C_L(G, D, \widetilde{C})$ is a $[33, 5, 27]$-code. Note that the parameters, that is, the information rate $R = 5/33$ and the relative minimum distance $\delta = 27/33$, lie above the Gilbert-Varshamov bound.

3. V-filtration and spectral numbers of hypersurface singularities

3.1. Introduction. The *Gauß-Manin connection* $\mathscr{G}$ is a *regular* $\mathscr{D}$-module associated to an isolated hypersurface singularity [32]. The *V-filtration* on $\mathscr{G}$ is defined by the $\mathscr{D}$-module structure. One can describe $\mathscr{G}$ in terms of integrals of holomorphic differential forms over vanishing cycles [1]. Classes of these differential forms in the *Brieskorn lattice* $\mathscr{H}''$ can be considered as elements of $\mathscr{G}$. The *V-filtration* on $\mathscr{H}''$ reflects the embedding of $\mathscr{H}''$ in $\mathscr{G}$ and determines the *singularity spectrum* which is an important invariant of the singularity.

E. Brieskorn [1] gave an algorithm to compute the *complex monodromy* based on the $\mathscr{D}$-module structure which is implemented in the computer algebra system Singular [17] in the library `mondromy.lib` [35]. In many respects, the *microlocal structure* of $\mathscr{G}$ and $\mathscr{H}''$ [32] seems to be more natural.

After a brief introduction to the theory of the *Gauß-Manin connection*, we describe how to use this structure for computing in $\mathscr{G}$ and give an explicit algorithm to compute the *V-filtration* on $\mathscr{H}''$. This also leads to a much more efficient algorithm to compute the *complex monodromy* and the singularity spectrum of an arbitrary isolated hypersurface singularity. All algorithms are implemented in the Singular library `gaussman.lib` [36] and are distributed with version 2.0.

For more theoretical background on this section see [37].

3.2. **Milnor fibration.** Let $f : X \to T$ be a *Milnor representative* [30] of an isolated hypersurface singularity $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with Milnor number μ. Then

$$f^{-1}(T') \cap X =: X' \xrightarrow{f} T' := T \setminus \{0\}$$

is a C^∞ fibre bundle with fibres $X_t := f^{-1}(t)$, $t \in T'$, of homotopy type of a bouquet of μ n-spheres.

The cohomology bundle $H^n := \bigcup_{t \in T'} H^n(X_t, \mathbb{C})$ is a flat complex vector bundle on T'. Hence, there is a natural flat connection on the sheaf $\mathscr{H}^n$ of holomorphic sections in H^n with covariant derivative $\partial_t : \mathscr{H}^n \to \mathscr{H}^n$. It induces a differential operator ∂_t on $(i_* \mathscr{H}^n)_0$ where $i : T' \hookrightarrow T$ denotes the inclusion.

Let $u : T^\infty \to T$, $\tau \mapsto \exp(2\pi i \tau)$, be the *universal covering* of T' and

$$X^\infty := X' \times_{T'} T^\infty$$

the *canonical Milnor fibre*. Then the natural maps $X_{u(\tau)} \cong X^\infty_\tau \hookrightarrow X^\infty$, $\tau \in T^\infty$, are homotopy equivalences. Hence, $H^n(X^\infty, \mathbb{C})$ can be considered as the space of global flat multivalued sections in H^n and as a trivial complex vector bundle on T^∞.

3.3. **Gauß-Manin connection.** There is a natural action of the fundamental group $\Pi_1(T', t)$, $t \in T'$, on $H^n(X_t, \mathbb{C}) \cong H^n(X^\infty, \mathbb{C})$. A positively oriented generator of $\Pi_1(T', t)$ operates via the *monodromy operator* M defined by

$$(Ms)(\tau) := s(\tau + 1)$$

for $s \in H^n(X^\infty, \mathbb{C})$ and $\tau \in T^\infty$. Let $M = M_s M_u$ be the decomposition of M into the semisimple part M_s and the unipotent part M_u, and set $N := \log M_u$. By the *monodromy theorem* [1, 24], the eigenvalues of M_s are roots of unity and $N^{n+1} = 0$. Let

$$H^n(X^\infty, \mathbb{C}) \cong \bigoplus_\lambda H^n(X^\infty, \mathbb{C})_\lambda$$

be the decomposition of $H^n(X^\infty, \mathbb{C})$ into the generalized eigenspaces of M,

$$H^n(X^\infty, \mathbb{C})_\lambda := \operatorname{Ker}(M_s - \lambda)\,, \qquad \lambda = \exp(-2\pi i \alpha)\,, \quad \alpha \in \mathbb{Q},$$

and let $M_\lambda := M\big|_{H^n(X^\infty, \mathbb{C})_\lambda}$. For $A \in H^n(X^\infty, \mathbb{C})_\lambda$, $\lambda = \exp(-2\pi i \alpha)$,

$$s(A, \alpha)(t) \;:=\; t^{\alpha - \frac{N}{2\pi i}} A(t) \;=\; t^\alpha \exp\Big(-\frac{N}{2\pi i} \log t\Big) A(t),$$

is monodromy invariant and defines a holomorphic section in H^n. The sections $i_* s(A, \alpha)$ span a ∂_t-invariant, finitely generated, free $\mathscr{O}_T[t^{-1}]$-submodule $\mathscr{G} \subset i_* \mathscr{H}^n$ of rank μ. Note that the direct image sheaf $i_* \mathscr{H}$ is in general not finitely generated. The *Gauß-Manin connection* is the *regular* $\mathbb{C}\{t\}[\partial_t]$-module $\mathscr{G}_0$, the stalk of $\mathscr{G}$ at 0 [1, 32].

3.4. **V-filtration.** Since $t^{\alpha - \frac{N}{2\pi i}}$ is invertible, $\psi_\alpha(A) := (i_* s(A, \alpha))_0$ defines an inclusion $\psi_\alpha : H^n(X^\infty, \mathbb{C}) \hookrightarrow \mathscr{G}_0$, satisfying the relations $\partial_t \psi_\alpha = \psi_{\alpha-1}\big(\alpha - \frac{N}{2\pi i}\big)$ and $t\psi_\alpha = \psi_{\alpha+1}$, by definition of $s(A, \alpha)$. This implies $(t\partial_t - \alpha)\psi_\alpha \;=\; \psi_\alpha\big(-\frac{N}{2\pi i}\big)$, $\exp(-2\pi i t\partial_t)\psi_\alpha = \psi_\alpha M_\lambda$, and the image

$$C_\alpha := \operatorname{im} \psi_\alpha \;=\; \operatorname{Ker}\,(t\partial_t - \alpha)^{n+1}$$

of ψ_α is the generalized α-eigenspace of $t\partial_t$. Moreover, $T : C_\alpha \to C_{\alpha+1}$ is bijective, and $\partial_t : C_\alpha \to C_{\alpha-1}$ is bijective for $\alpha \neq 0$. The *V-filtration* V on $\mathscr{G}_0$ is defined by

$$V^\alpha := V^\alpha \mathscr{G}_0 := \sum_{\alpha \leq \beta} \mathbb{C}\{t\} \cdot C_\beta\,, \quad V^{>\alpha} := V^{>\alpha}\mathscr{G}_0 := \sum_{\alpha < \beta} \mathbb{C}\{t\} \cdot C_\beta.$$

V^α and $V^{>\alpha}$ are free $\mathbb{C}\{t\}$-modules of rank μ with $V^\alpha / V^{>\alpha} \cong C_\alpha$.

3.5. **Brieskorn lattice.** The *Brieskorn lattice* [1] $\mathscr{H}'' := f_* \Omega_X^{n+1} / df \wedge d(f_* \Omega_X^{n-1})$ is a free $\mathscr{O}_T$-module of rank μ [38]. The *Gelfand-Leray form*

$$s\big([\omega]\big)(t) := \left[\frac{\omega}{df}\,\Big|_{X_t}\right] = \left[\operatorname{res}\Big(\frac{\omega}{f-t}\Big)\right]$$

defines a map $s : \mathscr{H}'' \to i_* \mathscr{H}^n$ with image in $\mathscr{G}$, inducing an isomorphism $\mathscr{H}''\big|_{T'} \cong \mathscr{H}^n$, and satisfying $\partial_t s\big([df \wedge \eta]\big) = s\big([d\eta]\big)$ by the *Leray residue formula* [1]. Since $\mathscr{H}_0'' = \Omega_{X,0}^{n+1} / df \wedge d\Omega_{X,0}^{n-1}$ is a torsion free $\mathbb{C}\{t\}$-module, $s : \mathscr{H}_0'' \hookrightarrow \mathscr{G}_0$ is an inclusion, and we identify $\mathscr{H}_0''$ with its image in $\mathscr{G}_0$. By a result of [29], we have $\mathscr{H}_0'' \subset V^{>-1}$.

3.6. **Microlocal structure.** The ring of *microdifferential operators* with constant coefficients

$$\mathbb{C}\{\{\partial_t^{-1}\}\} := \left\{ \sum_{i \geq 0} a_i \partial_t^{-i} \in \mathbb{C}[[\partial_t^{-1}]] \;\middle|\; \sum_{i \geq 0} \frac{a_i}{i!} t^i \in \mathbb{C}\{t\} \right\}$$

is a discrete valuation ring and $\mathbb{C}\{t\}$ is a free $\mathbb{C}\{\{\partial_t^{-1}\}\}$-module of rank 1. For $\alpha > -1$, $\partial_t t : C_\alpha \to C_\alpha$ is bijective and hence $t^{\partial_t t|_{C_\alpha}}$ is a $\mathbb{C}\{t\}$-automorphism of $\mathbb{C}\{t\} \cdot C_\alpha$ mapping the trivial $\mathbb{C}\{t\}[\partial_t t]$-structure to that of $\mathbb{C}\{t\} \cdot C_\alpha$. Hence, $\mathbb{C}\{t\} \cdot C_\alpha$ is a free $\mathbb{C}\{\{\partial_t^{-1}\}\}$-module of rank $\dim_{\mathbb{C}} C_\alpha$.

In particular, V^α, resp. $V^{>\alpha}$, is a free $\mathbb{C}\{\{\partial_t^{-1}\}\}$-module of rank μ for $\alpha > -1$, resp. $\alpha \geq -1$. Hence, $\mathscr{G}_0$ is a μ-dimensional vector space over the quotient field $\mathbb{C}(\partial_t^{-1}) := \mathbb{C}\{\{\partial_t^{-1}\}\}[\partial_t]$. Since $\partial_t^{-1} \mathscr{H}_0'' \subset \mathscr{H}_0''$ and $\mathscr{H}_0'' \subset V^{>-1}$, $\mathscr{H}_0''$ is a free $\mathbb{C}\{\{\partial_t^{-1}\}\}$-module of rank μ.

3.7. **Singularity spectrum.** The *Hodge filtration* F on $\mathscr{G}_0$ is defined by $F_k := F_k \mathscr{G}_0 := \partial_t^k \mathscr{H}''$. The *singularity spectrum* $\operatorname{Sp} : \mathbb{Q} \to \mathbb{N}$, defined by

$$\operatorname{Sp}(\alpha) := \dim_{\mathbb{C}} \operatorname{Gr}_V^\alpha \operatorname{Gr}_0^F \mathscr{G}_0$$

reflects the embedding of $\mathscr{H}_0''$ in $\mathscr{G}_0$ and satisfies the symmetry relation

$$\operatorname{Sp}(\alpha) = \operatorname{Sp}(n - 1 - \alpha)\,.$$

Since $\mathscr{H}_0'' \subset V^{>-1}$, this implies $V^{>-1} \supset \mathscr{H}_0'' \supset V^{n-1}$ or, equivalently, $\operatorname{Sp}(\alpha) = 0$ for $\alpha \leq -1$ or $\alpha \geq n$.

3.8. **Algorithm.** We abbreviate $\Omega := \Omega_{X,0}$, $\mathscr{H}'' := \mathscr{H}_0''$, $\mathscr{G} := \mathscr{G}_0$, $s := \partial_t^{-1}$, and we consider the rings $\mathbb{C}\{t\}$ and $\mathbb{C}\{\{s\}\}$. Then $[s^{-2}t, s] = 1$ and, hence, $t = s^2 \partial_s$.

3.8.1. *Idea.* Since $\partial_t t = s^{-1}t = s^{-1}s^2\partial_s = s\partial_s$, the

$$\mathscr{H}''_k := \sum_{j=0}^{k}(\partial_t t)^j \mathscr{H}'' = \sum_{j=0}^{k}(s\partial_s)^j \mathscr{H}''.$$

are $\mathbb{C}\{t\}$-lattices and $\mathbb{C}\{\{s\}\}$-lattices. Since $\mathscr{G}$ is regular, the *saturation*

$$\mathscr{H}''_\infty := \sum_{j=0}^{\infty}(\partial_t t)^j \mathscr{H}''$$

of $\mathscr{H}''$ is a $\mathbb{C}\{t\}$-lattice and, hence, $k_\infty := \min\{k \geq 0 \mid \mathscr{H}''_k = \mathscr{H}''_\infty\}$ is a finite number. For any $K \geq n+1$ the inclusions $V^{>-1} \supset \mathscr{H}'' \supset V^{n-1}$ imply inclusions

$$V^{>-1} \supset \mathscr{H}''_\infty \supset \mathscr{H}'' \supset \partial_t^{-1}\mathscr{H}'' \supset V^n \supset \partial_t^{-K}V^{>-1} \supset \partial_t^{-K}\mathscr{H}''_\infty$$

and $\mathscr{H}''_\infty$ and $\partial_t^{-K}\mathscr{H}''_\infty$ are $\partial_t t$-invariant. Hence, $\partial_t t$ and $t\partial_t$ induce endomorphisms $\overline{\partial_t t}, \overline{t\partial_t} \in \operatorname{End}_{\mathbb{C}}\big(\mathscr{H}''_\infty/\partial_t^{-K}\mathscr{H}''_\infty\big)$ such that the V-filtration $V_{\overline{t\partial_t}} = V^{\bullet+1}_{\overline{\partial_t t}}$ defined by $\overline{t\partial_t}$ on $\mathscr{H}''_\infty/\partial_t^{-K}\mathscr{H}''_\infty$ induces the V-filtration on the subquotient $\mathscr{H}''/\partial_t^{-1}\mathscr{H}'' = \operatorname{Gr}_0^F \mathscr{G}_0$.

3.8.2. *Computation.* By the *finite determinacy theorem*, we may assume that $f \in \mathbb{C}[\boldsymbol{x}]$, $\boldsymbol{x} := x_0, \dots, x_n$, is a polynomial. Since $\mathbb{C}[\boldsymbol{x}]_{(\boldsymbol{x})} \subset \mathbb{C}\{\boldsymbol{x}\}$ is faithfully flat and all data will be defined over $\mathbb{C}[\boldsymbol{x}]_{(\boldsymbol{x})}$, we may replace $\mathbb{C}\{\boldsymbol{x}\}$ by $\mathbb{C}[\boldsymbol{x}]_{(\boldsymbol{x})}$ and, similarly, $\mathbb{C}\{t\}$ by $\mathbb{C}[t]_{(t)}$ and $\mathbb{C}\{\{s\}\}$ by $\mathbb{C}[s]_{(s)}$ for the computation. With the additional assumption $f \in \mathbb{Q}[\boldsymbol{x}]$, all data will be defined over $\mathbb{Q}$, and we can apply methods of computer algebra. Using standard basis methods for *local rings*, $\mathbb{C}[\boldsymbol{x}]_{\langle\boldsymbol{x}\rangle}$ one can compute a monomial $\mathbb{C}$-basis $\boldsymbol{m} = (m_1, \dots, m_\mu)^T$ of

$$\Omega_f := \Omega^{n+1}/df \wedge \Omega^n \cong \mathbb{C}\{\boldsymbol{x}\}/(\partial_{\boldsymbol{x}} f).$$

Since $\mathscr{H}''/s\mathscr{H}'' \cong \Omega_f$, $\boldsymbol{m}$ represents a $\mathbb{C}\{\{s\}\}$-basis of $\mathscr{H}''$ and a $\mathbb{C}(s)$-basis of $\mathscr{G}$ by *Nakayama's lemma.*

The matrix A of t with respect to $\boldsymbol{m}$ is defined by $t\boldsymbol{m} =: A\boldsymbol{m}$. Since $t = s^2\partial_s$, we obtain for $g \in \mathbb{C}(s)^\mu$

$$tg\boldsymbol{m} = \big(gA + s^2\partial_s(g)\big)\boldsymbol{m}\,.$$

So the action of t in terms of the $\mathbb{C}(s)$-basis $\boldsymbol{m}$ is determined by the matrix A by the above formula.

A *reduced normalform* with respect to a *local monomial ordering* allows to compute the projection to the first summand in

$$\Omega_f \oplus df \wedge \Omega^n \twoheadrightarrow \Omega_f \oplus df \wedge \Omega^n / df \wedge d\Omega^{n-1} \cong \mathscr{H}''/s\mathscr{H}'' \oplus s\mathscr{H}''\ .$$

Since $t[\omega] = [f\omega]$ and $s^{-1}[df \wedge \eta] = \partial_t[df \wedge \eta] = [d\eta]$, the matrix A of t with respect to $\boldsymbol{m}$ can be computed up to arbitrarily high order.

The basis representation H''_k of $\mathscr{H}''_k$ with respect to $\boldsymbol{m}$ defined by $\mathscr{H}''_k =: H''_k\boldsymbol{m}$ can be computed inductively by

$$\begin{aligned}
\delta H_0 &:= H_0 := \mathbb{C}\{\{s\}\}^\mu,\\
\delta H_{k+1} &:= \operatorname{jet}_{-1}\big(s^{-1}\delta H_k \operatorname{jet}_k(A) + s\partial_s \delta H_k\big),\\
H_{k+1} &= H_k + \delta H_{k+1}.
\end{aligned}$$

Using standard basis methods, one can check if $H_k = H_{k+1}$ and compute a $\mathbb{C}\{\{s\}\}$-basis $\boldsymbol{m}'$ of $H_{k_\infty} =: H_\infty$ with

$$\delta(\boldsymbol{m}') := \max\bigl\{\operatorname{ord}\bigl(m'_{i_1,j_1}\bigr) - \operatorname{ord}\bigl(m'_{i_2,j_2}\bigr) \bigm| m'_{i_1,j_1} \neq 0 \neq m'_{i_2,j_2}\bigr\} \leq k_\infty.$$

Then the matrix A' of t with respect to the $\mathbb{C}\{\{s\}\}$-basis $\boldsymbol{m}'\boldsymbol{m}$ of $\mathscr{H}''_\infty$ is defined by the formula $\boldsymbol{m}'A + s^2\partial_s\boldsymbol{m}' =: A'\boldsymbol{m}'$, and $\operatorname{jet}_k(A') = \operatorname{jet}_k(A'_{\leq k})$ for $A'_{\leq k}$ defined by

$$\boldsymbol{m}' \operatorname{jet}_{k+\delta(\boldsymbol{m}')}(A) + s^2\partial_s\boldsymbol{m}' =: A'_{\leq k}\boldsymbol{m}'.$$

Hence, the basis representation of $\overline{\partial_t t} \in \operatorname{End}_{\mathbb{C}}\bigl(\mathscr{H}''_\infty/\partial_t^{-K}\mathscr{H}''_\infty\bigr)$ with respect to $\boldsymbol{m}'\boldsymbol{m}$ is

$$\overline{s^{-1}A' + s\partial_s} \;=\; \overline{s^{-1}A'_{\leq K} + s\partial_s} \;\in\; \operatorname{End}_{\mathbb{C}}\bigl(\mathbb{C}\{\{s\}\}^\mu/s^K\mathbb{C}\{\{s\}\}^\mu\bigr).$$

The basis representation H' of $\mathscr{H}''$ with respect to $\boldsymbol{m}'\boldsymbol{m}$ is defined by $H_0 =: H'\boldsymbol{m}'$, and $V^{\bullet+1}_{\overline{s^{-1}A'_{\leq K}+s\partial_s}}(H'/sH')\boldsymbol{m}'$ is the basis representation of $V(\mathscr{H}''/s\mathscr{H}'')$ with respect to $\boldsymbol{m}$. The matrix of $\overline{s^{-1}A'_{\leq K} + s\partial_s}$ with respect to the canonical $\mathbb{C}$-basis

$$\begin{pmatrix} 1 & & & s & & & s^2 & & & & s^{K-1} & & \\ & \ddots & & & \ddots & & & \ddots & & \cdots & & \ddots & \\ & & 1 & & & s & & & s^2 & & & & s^{K-1} \end{pmatrix}^t$$

of $\mathbb{C}\{\{s\}\}^\mu/s^K\mathbb{C}\{\{s\}\}^\mu$ is given by the block matrix

$$\begin{pmatrix} A'_1 & A'_2 & A'_3 & A'_4 & \cdots & A'_K \\ & A'_1+1 & A'_2 & A'_3 & \cdots & A'_{K-1} \\ & & A'_1+2 & A'_2 & \cdots & A'_{K-2} \\ & & & A'_1+3 & \ddots & \vdots \\ & & & & \ddots & A'_2 \\ & & & & & A'_1+K-1 \end{pmatrix}$$

where $A' = \sum_{k\geq 0} A'_k s^k$. Since the eigenvalues of A'_1 are rational, they can be computed using univariate factorization over the rational numbers. Then the V-filtration $V^{\bullet+1}_{\overline{s^{-1}A'_{\leq K}+s\partial_s}}$ can be computed using methods of linear algebra.

3.8.3. *Summary.* Here are the main steps of the algorithm:

1. Compute a $\mathbb{C}\{\{\partial_t^{-1}\}\}$-basis $\boldsymbol{m}$ of $\mathscr{H}''$.
2. For increasing k compute successively the lattices $\mathscr{H}''_k$ in terms of $\boldsymbol{m}$, and t in terms of $\boldsymbol{m}$ up to order k until $k = k_\infty$ and $\mathscr{H}''_k$ is the saturation $\mathscr{H}''_\infty$ of $\mathscr{H}''$.
3. Compute a $\mathbb{C}\{\{\partial_t^{-1}\}\}$-basis $\boldsymbol{m}'\boldsymbol{m}$ of $\mathscr{H}''_\infty$.
4. Compute t in terms of $\boldsymbol{m}$ up to order $\delta(\boldsymbol{m}') + n + 1$.
5. Compute t in terms of $\boldsymbol{m}'\boldsymbol{m}$ up to order $K := n + 1$.
6. Compute $\mathscr{H}''$ in terms of $\boldsymbol{m}'\boldsymbol{m}$.
7. Compute the V-filtration on $\mathscr{H}''_\infty/\partial_t^{-K}\mathscr{H}''_\infty$ in terms of $\boldsymbol{m}'\boldsymbol{m}$.
8. Compute the induced V-filtration on $\mathscr{H}''/\partial_t^{-1}\mathscr{H}''$ in terms of $\boldsymbol{m}'\boldsymbol{m}$.

3.8.4. *Example.* The SINGULAR library `gaussman.lib` [36] contains an implementation of the algorithm. We use it to compute an example. First, we have to load the library:

```
LIB "gaussman.lib";
```

We define the ring $R := \mathbb{Q}[x,y]_{(x,y)}$ and $f = x^5 + x^2y^2 + y^5 \in R$:

```
ring R=0,(x,y),ds;
poly f=x5+x2y2+y5;
```

Finally, we compute the V-filtration:

```
vfiltration(f);
==> [1]:  _[1]=-1/2  _[2]=-3/10  _[3]=-1/10  _[4]=0
==>       _[5]=1/10  _[6]=3/10   _[7]=1/2
==> [2]:  1,2,2,1,2,2,1
==> [3]:  [1]:  _[1]=gen(11)
==>       [2]:  _[1]=gen(10)  _[2]=gen(6)
==>       [3]:  _[1]=gen(9)   _[2]=gen(4)
==>       [4]:  _[1]=gen(5)
==>       [5]:  _[1]=gen(8)   _[2]=gen(3)
==>       [6]:  _[1]=gen(7)   _[2]=gen(2)
==>       [7]:  _[1]=gen(1)
==> [4]:  _[1]=y5  _[2]=y4  _[3]=y3   _[4]=y2   _[5]=xy
==>       _[6]=y   _[7]=x4  _[8]=x3   _[9]=x2  _[10]=x
==>      _[11]=1
==> [5]:  _[1]=2x2y+5y4  _[2]=2xy2+5x4  _[3]=x5-y5
==>       _[4]=y6
```

The result is a list with 5 entries: The first contains the spectral numbers, the second the corresponding multiplicities, the third $\mathbb{C}$-bases of the graded parts of the V-filtration on Ω_f in terms of the monomial $\mathbb{C}$-basis in the fourth entry, and the fifth a standard basis of the Jacobian ideal. A monomial $x^\alpha y^\beta$ in the fourth entry is considered as $x^\alpha y^\beta dx \wedge dy \in \Omega_f$.

As an application of the implementation, the third author could verify Hertling's conjecture [21] about the variance of the spectral numbers for all isolated hypersurface singularities of Milnor number ≤ 16.

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